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Stokes flow due to infinite arrays of stokeslets in three dimensions

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Abstract. Infinite periodic arrays of stokeslets in three dimensions are summed up by obtaining various rapidly converging infinite series. The three cases treated here are: 1. Identical stokeslets distributed at constant intervals on a line parallel to a plate, 2. An array of identical stokeslets distributed on a two-dimensional periodic lattice on a plane parallel to a plate, 3. The same array, but parallel to and in between two plates. Computational results are shown and comparisons with previously averaged expressions are made.

1. Introduction

In 1896 H.A. Lorentz [1], the famous Dutch physicist derived, among other things, the Stokes flow due to a point force in an infinite medium. He also derived a reciprocal theorem from which one also expresses the flow due to a point force above an infinite flat plate. The Stokes flow due to a point force now goes under the name of a stokeslet.

Since then, only a few additional analytic expressions for a stokeslet in different geometries have been found. The stokeslet for a point force inside or outside a sphere was given by Oseen [2], above an infinite plane again by Blake [3], between parallel plates by Liron & Mochon [4] and inside an infinite circular cylinder by Liron & Shahar [5].

Beginning with the work of Gray & Hancock [6], the role of the stokeslet as a basic building block in modeling slender body motion in Stokes flow, was established. Many works using distributions of stokeslets, and higher-order singularities ensued. These are nowadays used to model the various motions of flagella and cilia, suspensions, sedimentation problems and many more. For the use to flagellar motion see Lighthill [7,8], and for use in ciliary motion see the adaptation of Lighthill's theorem to cilia by Gueron & Liron [9,10].

Since cilia fields show quite frequently periodic, or almost periodic, motion, it follows that the basic building block to the resultant flows is an infinite, or doubly infinite regular array of stokeslets, all with the same strength, see Blake [11], Liron & Mochon [12], and Liron [13,14]. In all of the above papers these stokeslet fields were averaged to some extent, at least in one direction to yield a mean or averaged velocity. However, to be able to see local fluid behavior, this should not be done. This we accomplish in this paper, obtaining solutions without any averaging to the following problems: An infinite periodic, discrete, distribution of stokeslets on a line above a flat plate, Section 2; A doubly infinite array of such stokeslets above a flat plate, Section 3, and between parallel plates, Section 4. Except for the latter, these are infinite summations of the Lorentz solutions.

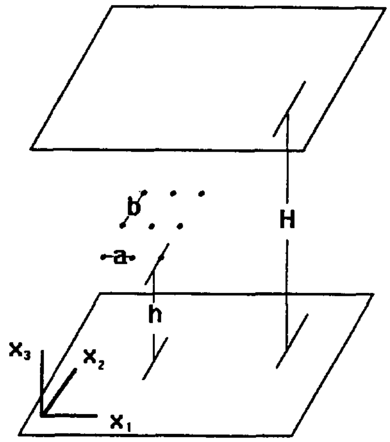


Fig. 1. Configuration of all three problems and the stokeslet distribution. For the third problem the plates are at $x_3 = 0, H$ and the stokeslets are located at height $x_3 = h$ above the lower plate. All stokeslets are identical. For the first two problems, the top plate does not exist.

2. An infinite line of discrete stokeslets above a flat plate

Consider an infinite array of identical stokeslets situated at a height $x_3 = h$ above a flat plate ($x_3 = 0$), as in Fig. 1 where we disregard the plate at $x_3 = H$, and assume that only *one* line of stokeslets exists, a distance a apart. The coordinate system is defined such that the stokeslets are situated at the points $\boldsymbol{\xi}_n = (na, 0, h), n = 0, \pm 1, \pm 2, \ldots$ Because of the periodicity of the problem, we can restrict our attention to the strip $0 \leq x_1 < a, -\infty < x_2 < +\infty, 0 \leq x_3 < \infty$.

For every $\mathbf{x} = (r_1^0, r_2, x_3)$ in this strip, the radius vector form the stokeslets at $\boldsymbol{\xi}_n$ to $\mathbf{x}$ is

$$\mathbf{r}_n = (r_1^0 - na, r_2, \alpha), n = 0, \pm 1, \ldots; \alpha = x_3 - h, \tag{2.1}$$

and from the image point to $\mathbf{x}$

$$\mathbf{R}_n = (r_1^0 - na, r_2, \beta), n = 0, \pm 1, \ldots; \beta = x_3 + h, \tag{2.2}$$

where

$$0 \leq r_1^0 < a. \tag{2.3}$$

If $G_j^k(\mathbf{x}, \boldsymbol{\xi})$, $j = 1, 2, 3$, is the velocity (Green's function) at $\mathbf{x}$ for a stokeslet situated at $\boldsymbol{\xi}$ pointing in the k direction ($k = 1, 2, 3$; Cartesian coordinates) and such that the no-slip condition is satisfied on the plane $x_3 = 0$, then

$$\begin{aligned} G_j^k &= \frac{1}{8\pi\mu}\left[\left\{\frac{\delta_{jk}}{r} + \frac{r_j r_k}{r^3}\right\} - \left\{\frac{\delta_{jk}}{R} + \frac{R_j R_k}{R^3}\right\}\right. \\ &\quad \left. +2\xi_3(\delta_{k\alpha}\delta_{\alpha l} - \delta_{k3}\delta_{3l})\frac{\partial}{\partial R_l}\left\{\frac{\xi_3 R_j}{R^3} - \left(\frac{\delta_{j3}}{R} + \frac{R_j R_3}{R^3}\right)\right\}\right], \\ &\quad k = 1, 2, 3; \quad j = 1, 2, 3, \end{aligned} \tag{2.4}$$

where $\alpha = 1, 2$ and $\mathbf{r}$ and $\mathbf{R}$ are defined as

$$\begin{aligned} \mathbf{r} &= (x_1 - \xi_1, x_2 - \xi_2, x_3 - \xi_3), \\ \mathbf{R} &= (x_1 - \xi_1, x_2 - \xi_2, x_3 + \xi_3), \\ r &= |\mathbf{r}|, R = |\mathbf{R}|. \end{aligned} \tag{2.5}$$

The pressure is given by

$$p^k = \frac{1}{4\pi}\left[\left\{\frac{r_k}{r} - \frac{R_k}{R^3}\right\} - 2\xi_3(\delta_{k\alpha}\delta_{\alpha l} - \delta_{k3}\delta_{3l})\frac{\partial}{\partial R_l}\left(\frac{R_3}{R^3}\right)\right], k = 1,2,3, \tag{2.6}$$

see Blake [3].

To obtain the velocity at x due to the infinite sequence of stokeslets at ξ_n, we have to sum

$$U_j^k = \sum_{n=-\infty}^{\infty} G_j^k(\mathbf{x}, \xi_n). \tag{2.7}$$

We now show some details of deriving expressions for U_j^k in (2.7). We first derive closed-form integrals, which we later convert to rapidly converging infinite series. The full expressions are given in Appendix A.

2.1. Obtaining expressions for U_j^k

Define

$$\rho_n^2 = (r_1^0 + na)^2 + r_2^2, \quad s^2 = r_2^2 + \alpha^2, \quad p^2 = r_2^2 + \beta^2. \tag{2.8}$$

Then

$$r_n = \sqrt{\rho_n^2 + \alpha^2}, \quad R_n = \sqrt{\rho_n^2 + \beta^2}.$$

To obtain U_1^1 we have

$$8\pi\mu U_1^1 = \sum_{n=-\infty}^{\infty} G_1^1 = \sum_{n=-\infty}^{\infty}\left[\left\{\frac{1}{r} - \frac{1}{R}\right\} + \left\{\frac{r_1^2}{r^3} - \frac{r_1^2}{R^3}\right\} + 2h\frac{\partial}{\partial R_1}\left\{\frac{hR_1}{R^3} - \frac{R_1R_3}{R^3}\right\}\right], \tag{2.9}$$

where for simplicity we have omitted the subscript n on all r, R.

To compute the first sum, use the Lipschitz integral (see G.N. Watson [15])

$$\frac{1}{(\rho^2 + a^2)^{1/2}} = \int_0^\infty J_0(\rho\lambda)e^{-|a|\lambda}\,\mathrm{d}\lambda, \tag{2.10}$$

to obtain

$$\begin{aligned} S_1 &= \sum_{n=-\infty}^{\infty}\left(\frac{1}{r_n} - \frac{1}{R_n}\right) = \sum_{n=-\infty}^{\infty}\int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]e^{-|r_1(n)|\lambda}\,\mathrm{d}\lambda \\ &= \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]\sum_{n=-\infty}^{\infty} e^{-|r_1(n)|\lambda}\,\mathrm{d}\lambda \\ &= \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]\frac{\cosh\left(\left(r_1^0 - \frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda. \end{aligned} \tag{2.11}$$

We can now use the following expression, see Liron & Mochon [4]:

$$\int_0^\infty J_0(bx)x^{v+1}F(x)\mathrm{d}x = \pi i \text{ (sum of residues in upper half plane of } F(z)z^{v+1}H_v^{(1)}(bz) \text{ including one half of the residue at } z = 0\text{)}, \tag{2.12}$$

when $F(z)$ is an even function of z and $F(z)$ decays exponentially to zero on the real axis, as $x = \operatorname{Re} z \to \pm\infty$, and b real, to obtain

$$S_1 = \frac{4}{a}\sum_{l=1}^{\infty} \cos\left(\frac{2\pi l r_1^0}{a}\right)\left[K_0\left(\frac{2\pi l}{a}s\right) - K_0\left(\frac{2\pi l}{a}p\right)\right] + \frac{2}{a}\ln\left(\frac{p}{s}\right). \tag{2.13}$$

Here K_0 is the modified Bessel function of the first kind of order zero, (see Abramowitz & Stegun [16]).

To obtain the second sum in (2.9), we write

$$S_2 = \sum_{n=-\infty}^{\infty}\left\{\frac{r_1^2}{r^3} - \frac{r_1^2}{R^3}\right\} = S_1 - \frac{\partial}{\partial r_1^0}\sum_{n=-\infty}^{\infty}\left\{\frac{r_1}{r} - \frac{r_1}{R}\right\}, \tag{2.14}$$

where

$$\sum_{n=-\infty}^{\infty}\left\{\frac{r_1}{r} - \frac{r_1}{R}\right\} = \sum_{n=-\infty}^{\infty} r_1(n)\int_0^{\infty}[J_0(s\lambda) - J_0(p\lambda)]e^{-|r_1(n)|\lambda}\mathrm{d}\lambda.$$

Changing the order of summation and integration we can write

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} r_1(n)e^{-|r_1(n)|\lambda} &= \sum_{n=0}^{\infty}(r_1^0 + na)e^{-(r_1^0+na)\lambda} - \sum_{n=1}^{\infty}(na - r_1^0)e^{-(na-r_1^0)\lambda} \\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\sum_{n=0}^{\infty} e^{-(r_1^0+na)\lambda} + \frac{\mathrm{d}}{\mathrm{d}\lambda}\sum_{n=1}^{\infty} e^{-(na-r_1^0)\lambda} \\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\sum_{n=0}^{\infty} e^{-(r_1^0+na)\lambda} - \sum_{n=1}^{\infty} e^{-(na-r_1^0)\lambda}\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{e^{-r_1^0\lambda}}{1-e^{-a\lambda}} - \frac{e^{-(a-r_1^0)\lambda}}{1-e^{-a\lambda}}\right] = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\sinh\left(\left(\frac{a}{2} - r_1^0\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\right],
\end{aligned}$$

and obtain

$$\sum_{n=-\infty}^{\infty}\left\{\frac{r_1}{r} - \frac{r_1}{R}\right\} = -\int_0^{\infty}[J_0(s\lambda) - J_0(p\lambda)]\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\sinh\left(\left(\frac{a}{2} - r_1^0\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\right]\mathrm{d}\lambda.$$

Integrate by parts and insert in (2.14) to obtain

$$S_2 = \sum_{n=-\infty}^{\infty}\left\{\frac{r_1^2}{r^3} - \frac{r_1^2}{R^3}\right\} = S_1 - \int_0^{\infty}[sJ_1(s\lambda) - pJ_1(p\lambda)]\frac{\lambda\cosh\left(\left(\frac{a}{2} - r_1^0\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\mathrm{d}\lambda.$$

We can now use (2.12) to transform the integral to an infinite series and the final form of S_2 is

$$S_2 = S_1 - \frac{8\pi}{a^2}\sum_{l=1}^{\infty} l\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[sK_1\left(\frac{2\pi l}{a}s\right) - pK_1\left(\frac{2\pi l}{a}p\right)\right]. \tag{2.15}$$

Again K_1 is the modified Bessel function of the first kind of order one.

For the last sum in (2.9) notice that

$$2h\frac{\partial}{\partial R_1}\left\{\frac{hR_1}{R^3}-\frac{R_1R_3}{R^3}\right\}=-2hx_3\frac{\partial}{\partial r_1}\frac{r_1}{R^3},\ \text{and}\ \frac{r_1}{R^3}=-\frac{\partial}{\partial r_1}\frac{1}{R},$$

so, using (2.13), we obtain

$$\begin{aligned}2h\frac{\partial}{\partial R_1}\left\{\frac{hR_1}{R^3}-\frac{R_1R_3}{R^3}\right\} &= -2hx_3\int_0^\infty J_0(p\lambda)\frac{\lambda^2\cosh\left(\left(r_1^0-\frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda \\ &= \frac{32hx_3\pi^2}{a^3}\sum_{l=1}^{\infty} l^2\cos\left(\frac{2\pi l r_1^0}{a}\right)K_0\left(\frac{2\pi l}{a}p\right). \qquad (2.16)\end{aligned}$$

This completes the computation of U_1^1.

To calculate the other components of velocity and pressure we need

$$\begin{aligned}S_3 &= \sum_{n=-\infty}^{\infty}\left\{\frac{r_2^2}{r^3}-\frac{r_2^2}{R^3}\right\}=-r_2^2\sum_{n=-\infty}^{\infty}\left[\frac{1}{z}\frac{\partial}{\partial z}\frac{1}{r}-\frac{1}{p}\frac{\partial}{\partial p}\frac{1}{R}\right] \\ &= r_2^2\int_0^\infty\lambda\left[\frac{J_1(s\lambda)}{s}-\frac{J_1(p\lambda)}{p}\right]\frac{\cosh\left(\left(r_1^0-\frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda, \qquad (2.17)\end{aligned}$$

where $z\equiv x_3$. In a similar way we obtain

$$S_4=\sum_{n=-\infty}^{\infty}\left\{\frac{\alpha^2}{r^3}-\frac{\beta^2}{R^3}\right\}=\int_0^\infty\lambda\left[\frac{\alpha^2J_1(s\lambda)}{s}-\frac{\beta^2J_1(p\lambda)}{p}\right]\frac{\cosh\left(\left(r_1^0-\frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda,$$

$$S_5=\sum_{n=-\infty}^{\infty}\left\{\frac{r_2r_1}{r^3}-\frac{r_2r_1}{R^3}\right\}=-r_2\int_0^\infty\lambda[J_0(s\lambda)-J_0(p\lambda)]\frac{\sinh\left(\left(r_1^0-\frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda,$$

$$S_6=\sum_{n=-\infty}^{\infty}\left\{\frac{r_3r_1}{r^3}-\frac{R_3r_1}{R^3}\right\}=-\int_0^\infty\lambda[\alpha J_0(s\lambda)-\beta J_0(p\lambda)]\frac{\sinh\left(\left(r_1^0-\frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda,$$

$$S_7=\sum_{n=-\infty}^{\infty}\left\{\frac{r_2r_3}{r^3}-\frac{r_2R_3}{R^3}\right\}=r_2\int_0^\infty\lambda\left[\frac{\alpha J_1(s\lambda)}{s}-\frac{\beta J_1(p\lambda)}{p}\right]\frac{\cosh\left(\left(r_1^0-\frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda.$$

All the other terms depend on $\sum 1/R^3$ or $\sum r_1/R^3$ which are parts of results above. All the above are then transformed to infinite series using (2.12). Complete expressions are given in Appendix A.

2.2. COMPUTATION CONSIDERATIONS AND NUMERICAL RESULTS

The infinite series in Appendix A all involve K_0 or K_1 and the arguments $l(2\pi p/a)$ or $l(2\pi s/a)$, where l is the running index. Since $K_0(x)$ and $K_1(x)$ both decay exponentially with $x \to \infty$, all series are rapidly converging except for p/a or s/a zero or very small. The smallest p can get is h, the height above the plate, and so vanishes only on the plate, but s is zero for $r_2 = \alpha = 0$, i.e., on the line $x_2 = 0$, $x_3 = h$, the line on which the stokeslets are distributed. In this case the expressions in the Appendix are invalid, and we have to calculate the infinite integrals. To accelerate convergence of the integrals, we proceed as follows. Take for example S_1

$$
\begin{aligned}
S_1 &= \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)\frac{\cosh\left(\left(r_1^0 - \frac{a}{2}\right)\lambda\right)}{\sinh\left(\frac{a\lambda}{2}\right)}\,\mathrm{d}\lambda \\
&= \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]\frac{\exp(-(a-r_1^0)\lambda) + \exp(-r_1^0\lambda)}{1-\exp(-a\lambda)}\,\mathrm{d}\lambda \\
&= \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]\frac{\exp(-(a-r_1^0)\lambda)}{1-\exp(-a\lambda)}\,\mathrm{d}\lambda \\
&\quad + \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]\frac{\exp(-r_1^0\lambda)}{1-\exp(-a\lambda)}\,\mathrm{d}\lambda \\
&= \frac{1}{\sqrt{s^2 + (r_1^0)^2}} - \frac{1}{\sqrt{p^2 + (r_1^0)^2}} + \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]\frac{\exp(-(a+r_1^0)\lambda)}{1-\exp(-a\lambda)}\,\mathrm{d}\lambda \\
&\quad + \frac{1}{\sqrt{s^2 + (a-r_1^0)^2}} \\
&\quad - \frac{1}{\sqrt{p^2 + (a-r_1^0)^2}} + \int_0^\infty [J_0(s\lambda) - J_0(p\lambda)]\frac{\exp(-(2a-r_1^0)\lambda)}{1-\exp(-a\lambda)},
\end{aligned}
$$

which ensures exponential decay of the integrand, as $\exp(-a\lambda)$, even for $s = 0$. This is, of course, leaving the closest stokeslets in their original form, and summing up the rest. This can be repeated to obtain faster convergence, if needed.

In Figs. 2a–d we show some examples of results. In Fig. 2a we show U_1^1 as a function of x_3 for a fixed $x_1(= 0.3)$ and various values of x_2. The line of computation closest to a stokeslet is $x_2 = 0$, and indeed U_1^1 achieves a maximum there for $x_3 = h = 0.5$, as expected. Fig. 2b shows U_3^3 under the same conditions, and Fig. 2c shows examples of the other components along a similar line. To demonstrate the behavior of the pressure distribution we depict P^3 for the same line as U_1^1, in Fig. 2d.

3. Infinite arrays of stokeslets above a flat plate

In the section we extend the results of the previous section to a doubly infinite array of stokeslets, as seen in Fig. 1, where we disregard the plate at $x_3 = H$. The stokeslets are spaced a distance a apart in the x_1 direction, a distance b apart in the x_2 direction, and at a height h, as before, above a flat plate.

(a)

(b)

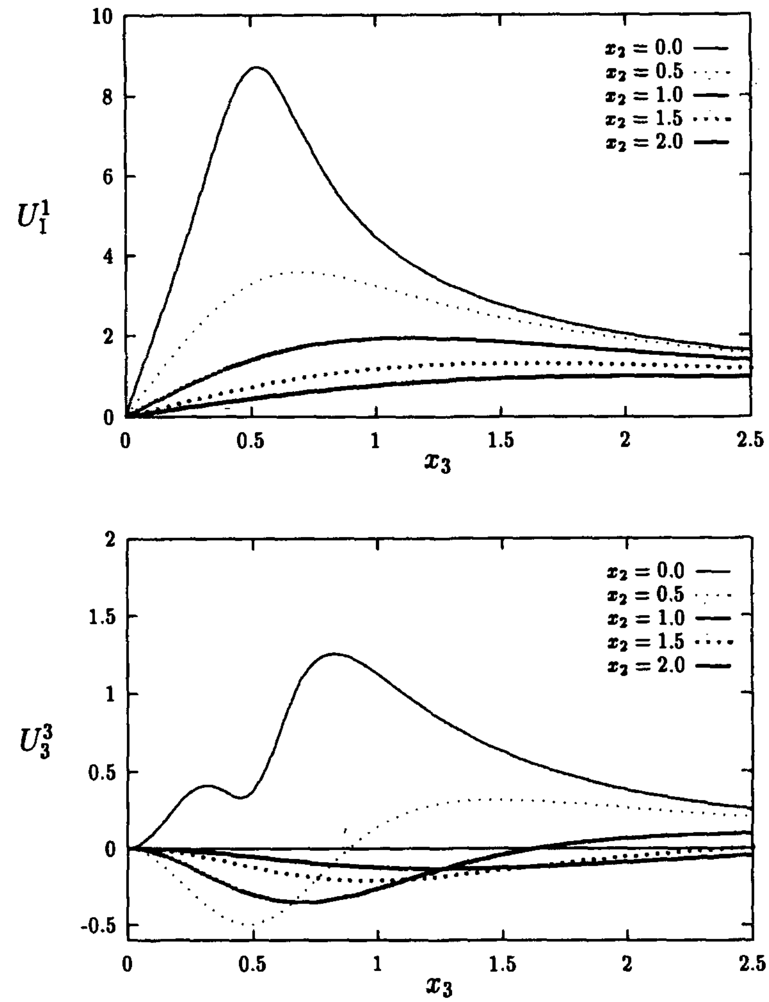


Fig. 2. Velocities and pressure due to a line of equally spaced identical stokeslets at height $h = 0.5$ above an infinite flat plate. The stokeslets are situated a distance a apart (see Appendix A for full details). (a) $8\pi U_1^1$ at $x_1 = 0.3$ as a function of x_3 for various values of x_2; (b) $8\pi U_3^3$ at $x_1 = 0.3$ as a function of x_3 for several values of x_2; (c) $8\pi[U_3^1, U_1^3, U_3^2, U_2^3]$ for $x_1 = x_2 = 0.3$ as a function of x_3. (d) $4\pi P^3$ at $x_1 = 0.3$ as a function of x_3 for several values of x_2.

Because of the periodicity of the problem we can restrict our attention to the strip $0 \leq x_1 < a, 0 \leq x_2 < b, 0 \leq x_3 < \infty$. Thus the stokeslets are situated at

$$\xi_{n,m} = (na, mb, h), n = 0, \pm 1, \dots m = 0, \pm 1 \dots,$$

and for any $\mathbf{x} = (r_1^0, r_2^0, x_3)$ in the above strip, the radius vector from $\xi_{n,m}$ to $\mathbf{x}$ is

$$\mathbf{r}_{n,m} = (r_1^0 - na, r_2^0 - mb, \alpha), \alpha = x_3 - h, \quad n = 0, \pm 1, \dots m = 0, \pm 1, \dots, \tag{3.1}$$

and from the image point

$$\mathbf{R}_{n,m} = (r_1^0 - na, r_2^0 - mb, \beta), \beta = x_3 + h, \quad n = 0, \pm 1, \dots m = 0, \pm 1, \dots, \tag{3.2}$$

where $0 \leq r_1^0 < a, 0 \leq r_2^0 < b$. As before $r_{n,m} = |\mathbf{r}_{n,m}|, R_{n,m} = |\mathbf{R}_{n,m}|$.

(c)

(d)

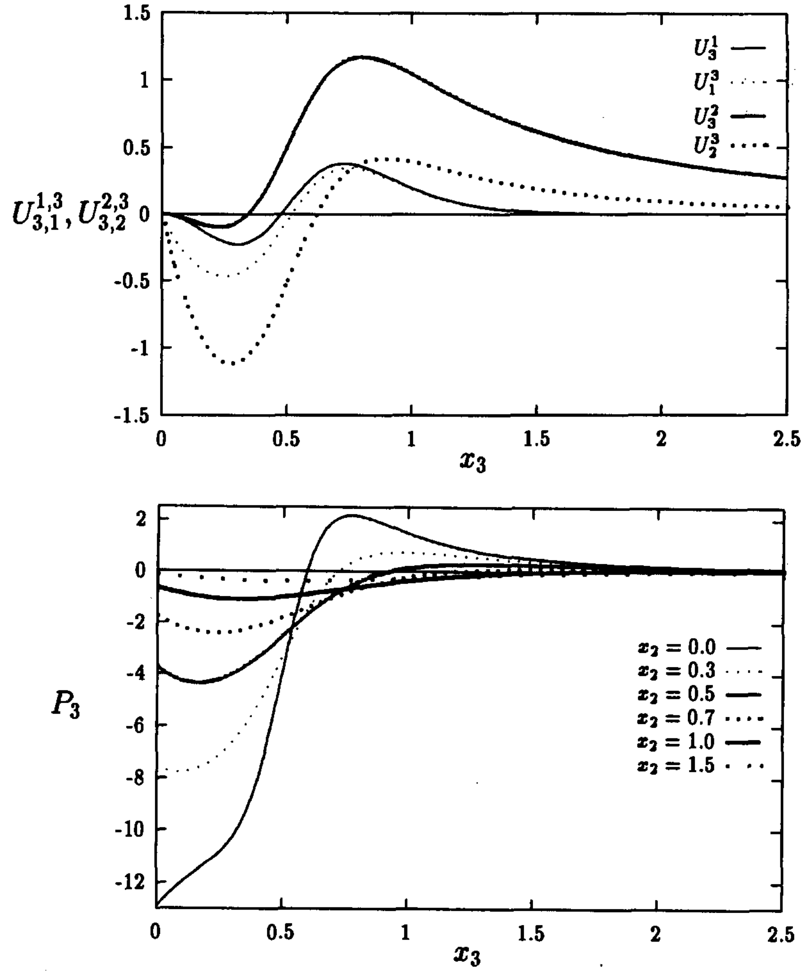


Fig. 2. Continued

To obtain the velocity due to the doubly infinite array of stokeslets at $\boldsymbol{\xi}_{n,m}$ we now have to sum

$$U_j^k = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_j^k(\mathbf{x}, \boldsymbol{\xi}_{n,m}), \tag{3.3}$$

where G_j^k is given in (2.5).

3.1. Obtaining expressions for U_j^k

Define

$$\rho_{n,m}^2 = (r_1^0 + na)^2 + (r_2^0 + mb)^2 = r_1^2(n) + r_2^2(m), \quad n = 0, \pm 1 \ldots; m = 0, \pm 1 \ldots$$
$$\rho_1^2 = r_1^2 + \alpha^2\,, \; \rho_2^2 = r_1^2 + \beta^2, \tag{3.4}$$

where in the second line we have omitted the explicit dependence on n. Thus

$$r_{n,m} = \sqrt{\rho_{n,m}^2 + \alpha^2}, \quad R_{n,m} = \sqrt{\rho_{n,m}^2 + \beta^2}. \tag{3.5}$$

We now show how to calculate S_1, similar to (2.13), except that we have a doubly infinite sum.

$$\begin{aligned}
S_1 &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\frac{1}{r_{n,m}}-\frac{1}{R_{n,m}}\right)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}[J_0(\rho_1\lambda)-J_0(\rho_2\lambda)]e^{-|r_2(m)|\lambda}\mathrm{d}\lambda\\
&= \sum_{n=-\infty}^{\infty}\int_0^{\infty}[J_0(\rho_1\lambda)-J_0(\rho_2\lambda)]\sum_{m=-\infty}^{\infty}e^{-|r_2(m)|\lambda}\,\mathrm{d}\lambda\\
&= \sum_{n=-\infty}^{\infty}\int_0^{\infty}[J_0(\rho_1\lambda)-J_0(\rho_2\lambda)]\frac{\cosh\left(\left(r_2^0-\frac{b}{2}\right)\lambda\right)}{\sinh\left(\frac{b\lambda}{2}\right)}\mathrm{d}\lambda\\
&= \sum_{n=-\infty}^{\infty}\left[\frac{4}{b}\sum_{l=1}^{\infty}\cos\left(\frac{2\pi l r_2^0}{b}\right)\left[K_0\left(\frac{2\pi l}{b}\rho_1\right)-K_0\left(\frac{2\pi l}{b}\rho_2\right)\right]+\frac{2}{b}\ln\left(\frac{\rho_2}{\rho_1}\right)\right]\\
&= \frac{4}{b}\sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}\cos\left(\frac{2\pi l r_2^0}{b}\right)\left[K_0\left(\frac{2\pi l}{b}\rho_1\right)-K_0\left(\frac{2\pi l}{b}\rho_2\right)\right]+\frac{2}{b}\ln\sum_{n=-\infty}^{\infty}\left(\frac{\rho_2}{\rho_1}\right).
\end{aligned}$$

To obtain

$$\sum_{n=-\infty}^{\infty}\ln\left(\frac{\rho_2}{\rho_1}\right)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\ln\left(\frac{\beta^2+r_1^2}{a^2+r_1^2}\right),\quad \text{let } |\alpha|=p\beta \text{ and define}$$

$$f(\beta)=\sum_{n=-\infty}^{\infty}\ln\left(\frac{\beta^2+(r_1^0+na)^2}{p^2\beta^2+(r_1^0+na)^2}\right),$$

from which the derivative is

$$f'(\beta)=\sum_{n=-\infty}^{\infty}\ln\left[\frac{2\beta}{r_1^2+\beta^2}-\frac{2p^2\beta}{r_1^2+p^2\beta^2}\right].$$

Using

$$\sum_{n=-\infty}^{\infty}\frac{1}{a^2+(n-b)^2}=\frac{\pi}{2a}\frac{\sinh(2\pi a)}{\sinh^2(\pi a)+\sin^2(\pi b)},$$

we have

$$\begin{aligned}
f'(\beta) &= \frac{\pi}{a}\left[\frac{\sinh\left(\frac{2\pi\beta}{a}\right)}{\sinh^2\left(\frac{\pi\beta}{a}\right)+\sin^2\left(\frac{\pi r_1^0}{a}\right)}-\frac{p\sinh\left(\frac{2\pi p\beta}{a}\right)}{\sinh^2\left(\frac{\pi p\beta}{a}\right)+\sin^2\left(\frac{\pi r_1^0}{a}\right)}\right]\\
&= \frac{\mathrm{d}}{\mathrm{d}\beta}\left\{\ln\left[\frac{\sinh^2\left(\frac{\pi\beta}{a}\right)+\sin^2\left(\frac{\pi r_1^0}{a}\right)}{\sinh^2\left(\frac{\pi p\beta}{a}\right)+\sin^2\left(\frac{\pi r_1^0}{a}\right)}\right]\right\}=\frac{\mathrm{d}}{\mathrm{d}\beta}F(\beta).
\end{aligned}$$

Thus $F(\beta)$ and $f(\beta)$ differ by a constant, at most.

For $\beta = 0, f(0) = 0, F(0) = 0$, for $r_1^0 \neq 0$, implying $F(\beta) = f(\beta)$. For $r_1^0 = 0, f(0) = \ln \frac{1}{p^2}$ and $F(0) = \ln \frac{1}{p^2}$. and again $F(\beta) = f(\beta)$. Thus

$$\sum_{n=-\infty}^{\infty} \ln\left(\frac{\beta^2 + r_1^2}{\alpha^2 + r_1^2}\right) = \ln\left[\frac{\sinh^2\left(\frac{\pi\beta}{a}\right) + \sin^2\left(\frac{\pi r_1^0}{a}\right)}{\sinh^2\left(\frac{\pi\alpha}{a}\right) + \sin^2\left(\frac{\pi r_1^0}{a}\right)}\right],$$

and this completes S_1 for the case $\rho_1^2(n) = r_1^2(n) + \alpha^2$ $\rho_2^2(n) = r_1^2(n) + \beta^2$ and $\rho_1^2(0) > 0$, namely

$$\begin{aligned} S_1 &= \frac{4}{b}\sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}\cos\left(\frac{2\pi l r_2^0}{b}\right)\left[K_0\left(\frac{2\pi l}{b}\rho_1\right) - K_0\left(\frac{2\pi l}{b}\rho_2\right)\right] \\ &+ \frac{2}{b}\ln\left[\frac{\sinh^2\left(\frac{\pi\beta}{a}\right) + \sin^2\left(\frac{\pi r_1^0}{a}\right)}{\sinh^2\left(\frac{\pi\alpha}{a}\right) + \sin^2\left(\frac{\pi r_1^0}{a}\right)}\right]. \end{aligned} \tag{3.6}$$

The expression for S_1 above becomes invalid for $\rho_1(0) = 0$, i.e., when $r_1^0 = 0$, $x_3 = h$. In this case we can repeat the steps leading to (3.6), interchanging the order of summation over m and n. This yields an expression similar to (3.6), namely

$$\begin{aligned} S_1 &= \frac{4}{a}\sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[K_0\left(\frac{2\pi l}{a}\rho_1\right) - K_0\left(\frac{2\pi l}{a}\rho_2\right)\right] \\ &+ \frac{2}{a}\ln\left[\frac{\sinh^2\left(\frac{\pi\beta}{b}\right) + \sin^2\left(\frac{\pi r_2^0}{b}\right)}{\sinh^2\left(\frac{\pi\alpha}{b}\right) + \sin^2\left(\frac{\pi r_2^0}{b}\right)}\right], \end{aligned} \tag{3.7}$$

for the case $\rho_1^2(m) = r_2^2(m) + \alpha^2$, $\rho_2^2(m) = r_2^2(m) + \beta^2$ and $\rho_1^2(0) > 0$.

Expression (3.7) for S_1 above becomes invalid for $r_2^0 = 0$, $x_3 = h$. Notice that

(i) Either (3.6) or (3.7) is always valid. Both are invalid only for $r_1^0 = r_2^0 = 0, x_3 = h$, which is at a stokeslet.

(ii) The general term in the infinite series has $K_0(Cl\rho)$ as a multiplier where C is $\frac{2\pi}{a}$ or $\frac{2\pi}{b}$ and $\rho = \rho_1$ or ρ_2. Thus the argument is proportional both to l and $n (\rho \sim n)$. Since K_0 decays exponentially the series $\sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}()$ decays very rapidly.

(iii) For $\alpha = 0$, i.e., $x_3 = h$, use (3.7) if r_1^0 is also close to zero, and use (3.6) if r_2^0 is close to zero. If both r_1^0 and r_2^0 are close to zero then we are near a stokeslet and convergence is poor as both series blow up at $r_1^0 = r_2^0 = 0$.

We shall demonstrate one more component. The complete expressions, covering all cases, are given in Appendix B.

To obtain

$$S_2 = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left\{\frac{r_1^2}{r^3} - \frac{r_1^2}{R^3}\right\}, \tag{3.8}$$

use (2.15) to get

$$S_2 = S_1 - \frac{8\pi}{a^2} \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) \left[\rho_1 K_1\left(\frac{2\pi l}{a}\rho_1\right) - \rho_2 K_1\left(\frac{2\pi l}{a}\rho_2\right)\right], \tag{3.9}$$

where S_1 is given in (3.6), with ρ_1, ρ_2 as defined there.

Alternatively, write

$$\frac{1}{r^3} - \frac{1}{R^3} = -\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\left(\frac{1}{r}\right) + \frac{1}{\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{R}\right)$$

and

$$\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\left(\frac{1}{r}\right) = -\frac{1\partial}{\alpha\partial\alpha}\int_0^\infty J_0(\rho_1\lambda)e^{-|r_2|\lambda}\,\mathrm{d}\lambda = -\frac{1}{\rho_1}\int_0^\infty \lambda J_1(\rho_1\lambda)e^{-|r_2|\lambda}\,\mathrm{d}\lambda.$$

Thus

$$\begin{aligned} S_2 &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left\{\frac{r_1^2}{r^3} - \frac{r_1^2}{R^3}\right\} \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} r_1^2 \int_0^\infty \lambda\left(\frac{J_1(\lambda\rho_1)}{\rho_1} - \frac{J_1(\lambda\rho_2)}{\rho_2}\right) e^{-|r_2|\lambda}\,\mathrm{d}\lambda \\ &= \sum_{n=-\infty}^{\infty} r_1^2 \int_0^\infty \lambda\left(\frac{J_1(\lambda\rho_1)}{\rho_1} - \frac{J_1(\lambda\rho_2)}{\rho_2}\right) \frac{\cosh\left(\left(\frac{b}{2} - r_2^0\right)\lambda\right)}{\sinh\left(\frac{b\lambda}{2}\right)}\,\mathrm{d}\lambda \\ &= \frac{\pi}{ab}\left[\frac{\beta\sinh\left(\frac{2\pi\beta}{a}\right)}{\sinh^2\left(\frac{\pi\beta}{2}\right) + \sin^2\left(\frac{\pi r_1^0}{a}\right)} - \frac{a\sinh\left(\frac{2\pi\alpha}{a}\right)}{\sinh^2\left(\frac{\pi\alpha}{a}\right) + \sin^2\left(\frac{\pi r_1^0}{a}\right)}\right] \\ &\quad + \frac{8\pi}{b^2}\sum_{n=-\infty}^{\infty} r_1^2 \sum_{l=1}^{\infty} l\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[\frac{1}{\rho_1}K_1\left(\frac{2\pi l}{a}\rho_1\right) - \frac{1}{\rho_2}K_1\left(\frac{2\pi l}{a}\rho_2\right)\right], \end{aligned} \tag{3.10}$$

where, as before, ρ_1 and ρ_2 are defined in (3.4).

All expressions for the velocity and pressure, in summation, form, are given in Appendix B.

3.2. Numerical results

We show here a selection of results of computing velocities and pressure inside the grid $0 < x_1 \leq a, 0 \leq x_2 < b, 0 < x_3 < \infty$. In Fig. 3 we show the graphs of two of the three major components of the velocity U_1^1, and U_3^3 for a grid $a = b = 1, h = 0.5$, at $x_1 = 0.3$ and several values of x_2. The values of U_1^1 and U_2^2 are quite similar with U_2^2 showing less variations, and is not reproduced here. They are not too far from the corresponding kernel H of Liron & Mochon [12], or the kernel K of Blake [11]. The kernel H is the average of U_j^i in the x_2 direction, and the kernel K is averaging in the x_1 direction as well. These results seem to be quite independent of the relative size of the grid, and the height, if we are not close to a stokeslet, as can be seen in Fig. 4, where we vary a. Varying b or h we obtain similar variations as in Fig.

(a)

(b)

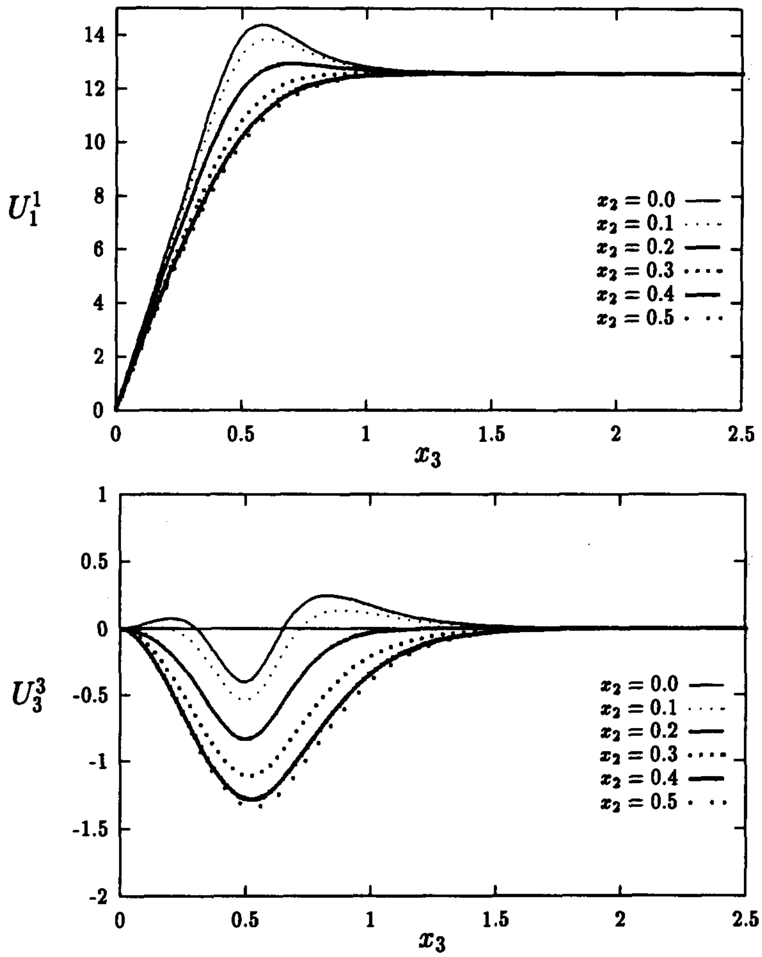


Fig. 3. Two of the three main components of velocity for the case $a = b = 1$, $h = 0.5$, as a function of x_3 for $x_1 = 0.3$ and several values of x_2. To be compared with the Kernels H and K, see text. (a) $8\pi U_1^1$, and (b) $8\pi U_3^3$.

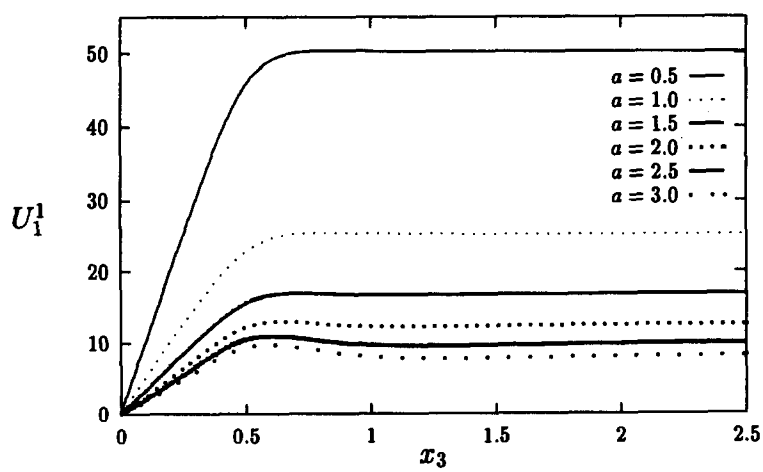


Fig. 4. $8\pi U_1^1$ as a function of x_3, for $x_1 = x_2 = 0.3$, $h = 0.5$, $b = 1$ and varying a.

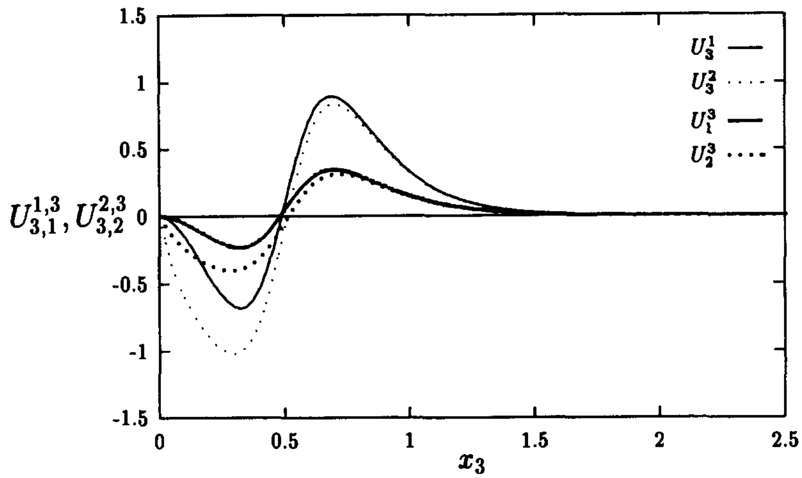


Fig. 5. $8\pi U^i_j, i \neq j$, as a function of x_3, for $x_1 = 0.3$, $x_2 = 0.1$, $a = b = 1$, $h = 0.5$.

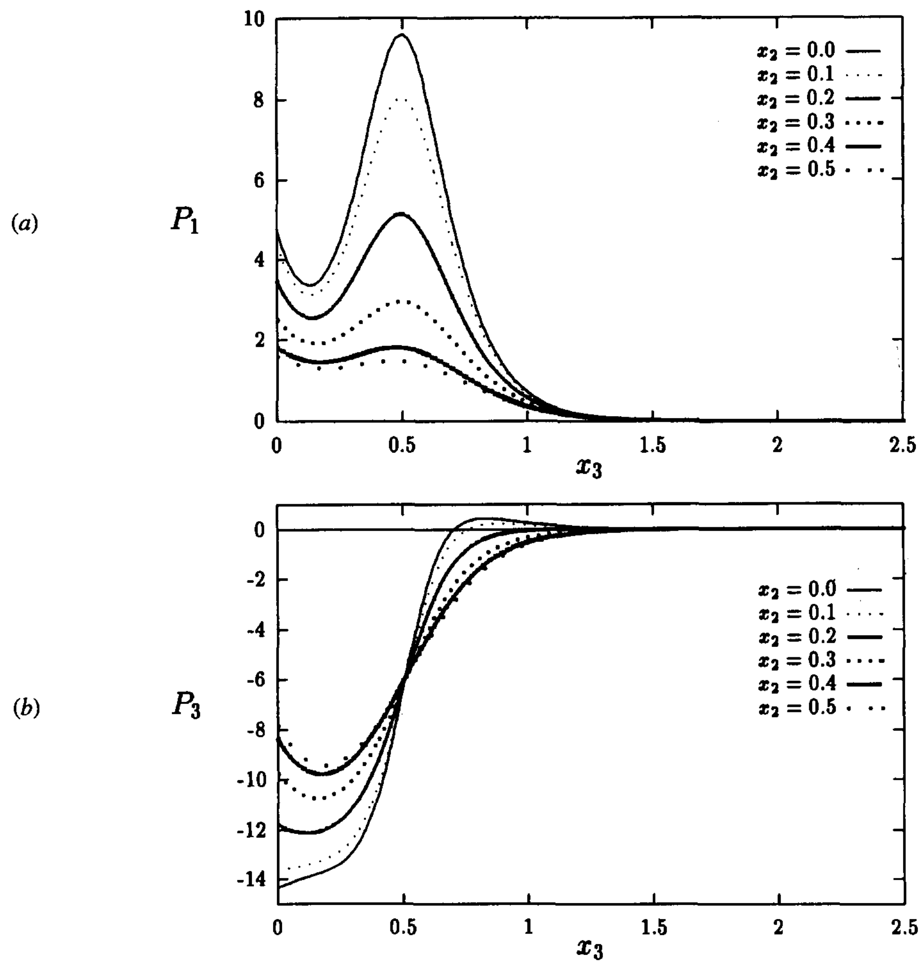


Fig. 6. The pressure as a function of x_3, for $x_1 = 0.3$, and several values of x_2. $h = 0.5$, $a = b = 1$. (a) $4\pi P_1$, (b) $4\pi P_3$.

4, and all of them turn out to be almost identical when scaled by the varying parameter. Fig. 3b, for U_3^3 is similar to H_3^3 and differs from U_1^1, U_2^2 mainly due to the fact that the flux in the x_3 direction is zero, contrary to the flux parallel to the plate (x_1, x_2 directions), see section 3 of Liron [13], dealing with flux and pressure due to stokeslets. If we take a point closer to a stokeslet, then the behavior changes, again similar to the corresponding H kernels, see Fig. 2 in Liron & Mochon [12]. This is rather surprising, considering the distance between the stokeslets seems to be large compared to the height above the plate (2:1). The behavior of the other components of the velocity are depicted in Fig. 5. The pressures are shown in Figs. 6, to complete the picture. P_2 is similar to P_1 in general features, and is not reproduced here.

4. Infinite arrays of stokeslets between parallel plates

We now consider the same problem as in Section 3, but with the stokeslets situated between two parallel flat plates of a distance H apart, see Fig. 1. As before the stokeslets are at height h above the bottom plate and distributed as in Section 3, at the points $\xi_{n,m} = (na, mb, h)$, $-\infty < n, m < \infty$. Liron & Mochon [4] gave a complete solution for a single stokeslet between parallel plates. This solution is

$$u_j^k = v_j^k + w_j^k, \quad j = 1,2,3; \qquad k = 1,2,3, \tag{4.1}$$

$$p^k = q^k + S^k, \quad k = 1,2,3. \tag{4.2}$$

As in (2.5) k is the direction of action of the force singularity $\mathbf{u}^k$ is the velocity vector, and p^k the corresponding pressure. The expressions for the velocity and pressure are,

$$\begin{aligned}4\pi\mu v_j^k = {}& \delta_{jk}\int_0^\infty J_0(\lambda\rho)\,\frac{\sinh(\lambda h)}{\sinh(\lambda H)}\,\sinh(\lambda(H-x_3))\,\mathrm{d}\lambda \\ &+\delta_{j\alpha}\delta_{k\beta}\frac{r_\alpha r_\beta}{\rho}\int_0^\infty \lambda J_1(\lambda\rho)\,\frac{\sinh(\lambda h)}{\sinh(\lambda H)}\,\sinh(\lambda(H-x_3))\,\mathrm{d}\lambda \\ &+\mathrm{sgn}(x_3-h)(\delta_{j3}\delta_{k\alpha}+\delta_{j\alpha}\delta_{k3})r_\alpha\int_0^\infty \lambda J_0(\lambda\rho)\frac{\sinh(\lambda h)}{\sinh(\lambda H)}\cosh(\lambda(H-x_3))\,\mathrm{d}\lambda,\end{aligned} \tag{4.3}$$

$$\begin{aligned}2\pi q^k = {}& \frac{r_\alpha}{\rho}\,\delta_{k\alpha}\int_0^\infty \lambda J_1(\lambda\rho)\,\frac{\sinh(\lambda h)}{\sinh(\lambda H)}\,\sinh(\lambda(H-x_3))\,\mathrm{d}\lambda \\ &+\mathrm{sgn}(x_3-h)\delta_{k3}\int_0^\infty \lambda J_0(\lambda\rho)\frac{\sinh(\lambda h)}{\sinh(\lambda H)}\cosh(\lambda(H-x_3))\,\mathrm{d}\lambda,\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}4\pi\mu\omega_\alpha^\beta &= -\frac{\mathrm{d}}{\mathrm{d}r_\beta}\frac{r_\alpha}{\rho}\int_0^\infty \xi J_1(\rho\xi)A_1(\xi)\mathrm{d}\xi, \\ 4\pi\mu\omega_3^3 &= \int_0^\infty J_0(\rho\xi)A_2(\xi)\mathrm{d}\xi, \\ 4\pi\mu\omega_{3,\alpha}^{\alpha,3} &= -\frac{\mathrm{d}}{\mathrm{d}r_\alpha}\int_0^\infty J_0(\rho\xi)[A_3(\xi)+A_4(\xi)]\mathrm{d}\xi,\end{aligned} \tag{4.5}$$

$$2\pi S^k = \int_0^\infty \left[\frac{r_\alpha}{\rho} J_1(\rho\xi)A_5(\xi)\delta_{k\alpha} + J_0(\rho\xi)A_6(\xi)\delta_{k3}\right] \mathrm{d}\xi. \tag{4.6}$$

Here α, β take on the values 1 or 2 only. The above expressions are for $x_3 > h$. For $x_3 < h$ replace x_3 by $H - x_3$ and h by $H - h$ under the integral signs only, (Expressions for A_1, A_2, A_4, A_5, A_6 are given in Appendix C), and $\rho^2 = r_1^2 + r_2^2$.

As before, $\mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the location of the stokeslet. For the solution to the flow due to the infinite array, we have to sum over all stokeslets,

$$U_j^k = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} u_j^k, \tag{4.7}$$

$$P^k = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} p^k. \tag{4.8}$$

4.1. Obtaining expressions for U_j^k

We shall demonstrate the technique by calculating U_1^1. As in Liron & Mochon [4] we use the fact that the integral

$$\int_0^\infty \lambda J_0(\rho\lambda) f(\lambda)\,\mathrm{d}\lambda$$

can be looked upon as the (inverse) zero-order Hankel transform. Also the double Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(r_1, r_2, x_3) e^{i(\lambda_1 r_1 + \lambda_2 r_2)}\,\mathrm{d}r_1\,\mathrm{d}r_2$$

equals to the zero-order Hankel transform if $\varphi(r_1, r_2, x_3) = \varphi_1[(r_1^2 + r_2^2)^{\frac{1}{2}}, x_3] = \varphi_1(\rho, x_3)$, see Sneddon [17].

Thus for the first term of (4.3) we get

$$\begin{aligned}\int_0^\infty J_0(\lambda\rho) f(\lambda)\,\mathrm{d}\lambda &= \int_0^\infty \lambda J_0(\lambda\rho)\frac{f(\lambda)}{\lambda}\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\lambda)}{\lambda} e^{i(\lambda_1 r_1 + \lambda_2 r_2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2,\end{aligned}$$

(see Appendix C for the definition of $f(\lambda)$ here). Using the above relation to sum the first term in (4.3), we obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_0^\infty J_0(\lambda\rho) f(\lambda)\,\mathrm{d}\lambda &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\lambda)}{\lambda}e^{i(\lambda_1 r_1 + \lambda_2 r_2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2 \\ &= \frac{\kappa_1}{2\pi}\sum_{n=-\infty}^{\infty} e^{i\kappa_1 r_1^0}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\sqrt{(\kappa_1 n)^2 + \lambda^2})}{\sqrt{(\kappa_1 n)^2 + \lambda^2}} e^{i\lambda_2 r_2}\,\mathrm{d}\lambda_2 \\ &= \frac{\kappa_1\kappa_2}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{f(\sqrt{(\kappa_1 n)^2 + (\kappa_2 m)^2}\ )}{\sqrt{(\kappa_1 n)^2 + (\kappa_2 m)^2}} e^{i(\kappa_1 n r_1^0 + \kappa_2 m r_2^0)}\end{aligned}$$

$$= \frac{\kappa_1\kappa_2}{2\pi}\left\{\frac{h(H-x_3)}{H} + 2\sum_{m=1}^{\infty}\frac{f(\kappa_2 m)}{(\kappa_2 m)}\cos(\kappa_2 m r_2^0) + 2\sum_{n=1}^{\infty}\frac{f(\kappa_1 n)}{(\kappa_1 n)}\cos(\kappa_1 n r_1^0)\right.$$

$$\left. = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{f(\sqrt{(\kappa_1 n)^2+(\kappa_2 m)^2}\ )}{\sqrt{(\kappa_1 n)^2+(\kappa_2 m)^2}}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right\}. \tag{4.9}$$

Here $\kappa_1 = 2\pi/a$; $\kappa_2 = 2\pi/b$. For the second term in (4.3)

$$\frac{r_1^2}{\rho}\int_0^{\infty}\lambda J_1(\lambda\rho)f(\lambda)\,\mathrm{d}\lambda = -r_1\frac{\mathrm{d}}{\mathrm{d}r_1}\int_0^{\infty}\lambda J_0(\lambda\rho)\frac{f(\lambda)}{\lambda}\,\mathrm{d}\lambda,$$

and then

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{r_1^2}{\rho}\int_0^{\infty}\lambda J_1(\lambda\rho)f(\lambda)\,\mathrm{d}\lambda = -\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} r_1\frac{\mathrm{d}}{\mathrm{d}r_1}\int_0^{\infty}\lambda J_0(\lambda\rho)\frac{f(\lambda)}{\lambda}\,\mathrm{d}\lambda$$

$$= -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} r_1\frac{\mathrm{d}}{\mathrm{d}r_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\lambda)}{\lambda}e^{i(\lambda_1 r_1+\lambda_2 r_2}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2$$

$$= -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} i\lambda_1 r_1\frac{f(\lambda)}{\lambda}e^{i(\lambda_1 r_1+\lambda_2 r_2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\lambda_2 r_2}\,\mathrm{d}\lambda_2\int_{-\infty}^{\infty}e^{i\lambda_1 r_1}\left[\frac{\lambda_1^2}{\lambda^2}f'(\lambda)+\frac{\lambda_2^2}{\lambda^3}f(\lambda)\right]\mathrm{d}\lambda_1$$

$$= \frac{\kappa_1\kappa_2}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\frac{(\kappa_1 n)^2}{(sq)^2}f'(sq)+\frac{(\kappa_2 m)^2}{(sq)^3}f(sq)\right]\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)$$

$$= \frac{\kappa_1\kappa_2}{2\pi}\left\{\frac{h(H-x_3)}{H}+2\sum_{m=1}^{\infty}\frac{f(\kappa_2 m)}{\kappa_2 m}\cos(\kappa_2 m r_2^0)+2\sum_{n=1}^{\infty}f'(\kappa_1 n)\cos(\kappa_1 n r_1^0)\right.$$

$$\left.+4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{(\kappa_1 n)^2}{(sq)^2}f'(sq)+\frac{(\kappa_2 m)^2}{(sq)^3}f(sq)\right]\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right\}. \tag{4.10}$$

Here $\lambda = \sqrt{\lambda_1^2+\lambda_2^2}$, $f(\lambda)$ is given in Appendix C, $\kappa_1 = \frac{2\pi}{a}$, $\kappa_2 = \frac{2\pi}{b}$ and $sq = ((\kappa_1 n)^2 + (\kappa_2 m)^2)^{1/2}$, inside the sums. In a similar way we obtain ω_1^1, in (4.5),

$$-\frac{\partial}{\partial r_1}\frac{r_1}{\rho}\int_0^{\infty}\lambda J_1(\rho\lambda)A_1(\lambda)\,\mathrm{d}\lambda = \frac{\partial^2}{\partial r_1^2}\int_0^{\infty}\lambda J_0(\rho\lambda)\frac{A_1(\lambda)}{\lambda}\,\mathrm{d}\lambda$$

$$= \frac{\partial^2}{\partial r_1^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A_1(\lambda)}{\lambda}e^{i(\lambda_1 r_1+\lambda_2 r_2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lambda_1^2\frac{A_1(\lambda)}{\lambda}e^{i(\lambda_1 r_1+\lambda_2 r_2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2.$$

and then

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lambda_1^2\frac{A_1(\lambda)}{\lambda}e^{i(\lambda_1 r_1+\lambda_2 r_2)}\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2$$

$$= -\frac{\kappa_1\kappa_2}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(\kappa_1 n)^2\frac{A_1(sq)}{sq}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)$$

$$= -\frac{\kappa_1\kappa_2}{2\pi}\left\{\frac{6h(H-h)x_3(H-x_3)}{H^3} + 2\sum_{n=1}^{\infty}(\kappa_1 n)A_1(\kappa_1 n)\cos(\kappa_1 n r_1^0)\right.$$

$$\left. +4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\kappa_1 n)^2\frac{A_1(sq)}{sq}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right\}. \qquad (4.11)$$

Expressions (4.9), (4.10), (4.11) complete the solution for U_1^1. Notice the free term in (4.11), which is a Poiseuille flow (parabolic profile). One can, of course, add an arbitrary Poiseuille flow to U_1^1, without contradicting the periodicity condition. The particular Poiseuille flow obtained in (4.11) is such as to ensure that the total flux in the x_1-direction is zero, since we have summed stokeslets, each of which creates zero flux. All components of the velocity and pressure are given in Appendix C.

4.2. Numerical results

In Figs. 7 we show $4\pi U_1^1$ for the same array as Section 3, but with another plate at $H = 1$. The additional plate now changes the picture completely as compared to the one plate case. The graphs are different to begin with because here the flux is always zero, see Liron [13], but more important the graphs change in size, shape and sign, when parameters are changed, which only act to rescale in the one plate case. These should be compared with the kernel D in Liron [13] which one obtains by averaging the present solution over x_2. This means that a much denser net must be used if we wish to obtain the averaged-out effect of all stokeslets. Evidently, this should be much denser than the one obviously sufficient for the previous case. Fig. 7a shows variations due to x_2 far from any stokeslet. Fig. 7b shows the change in behavior due to x_1, and Fig. 7c the change in behavior due to changes in h. It is particularly worthwhile noticing Fig. 7b, in which, at all values of x_1, for all x_3, $4\pi U_1^1$ is *negative*. This is impossible

(*a*)

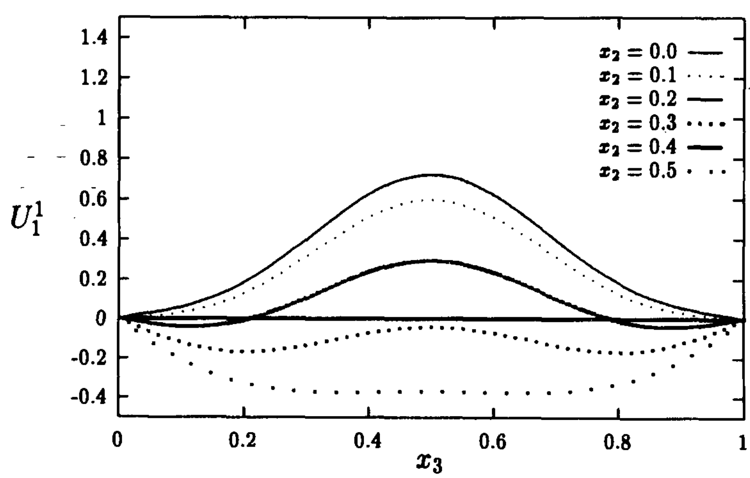
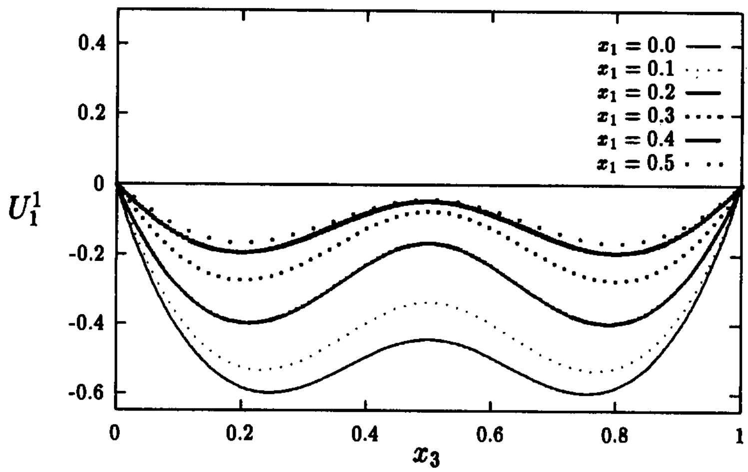


Fig. 7. $4\pi U_1^1$ as a function of x_3 for various values of the other variables and parameters. (a) $a = b = H = 1$, $h = 0.5, x_1 = 0.5$ and several x_2; (b) $a = b = H = 1, h = 0.5, x_2 = 0.3$ and various x_1; (c) $a = b = H = 1$, $x_1 = x_2 = 0.3$ and various h. Notice the large changes in behavior.

(*b*)



(*c*)

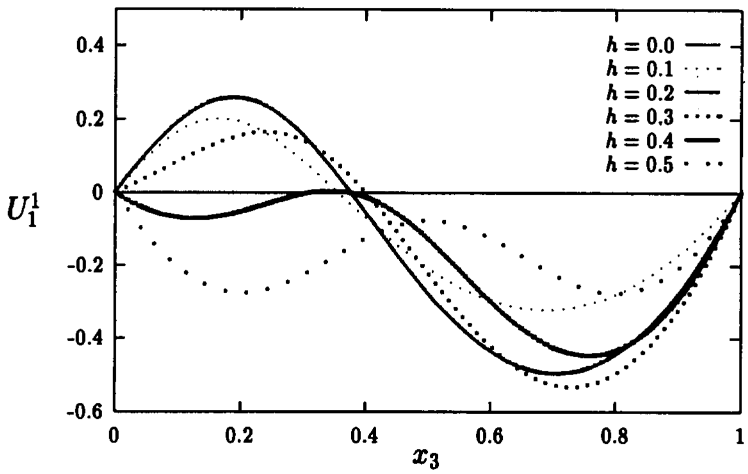


Fig. 7. Continued

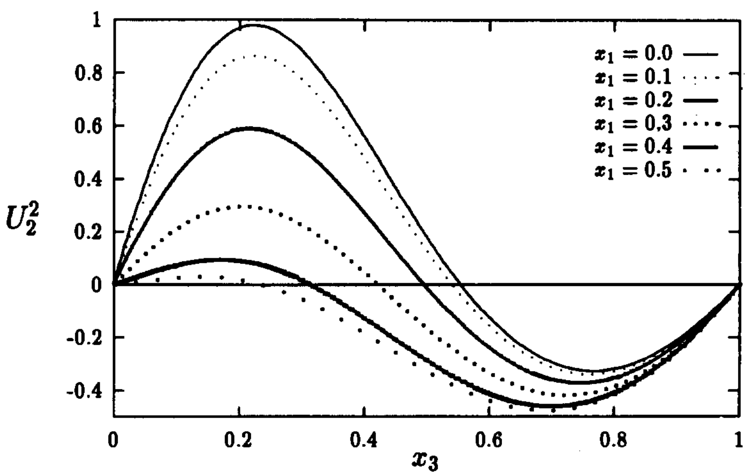


Fig. 8. $4\pi U_2^2$ as a function of x_3, for $a = b = H = 1$, $h = 0.2$, $x_2 = 0.5$ and several values of x_1.

(a)

(b)

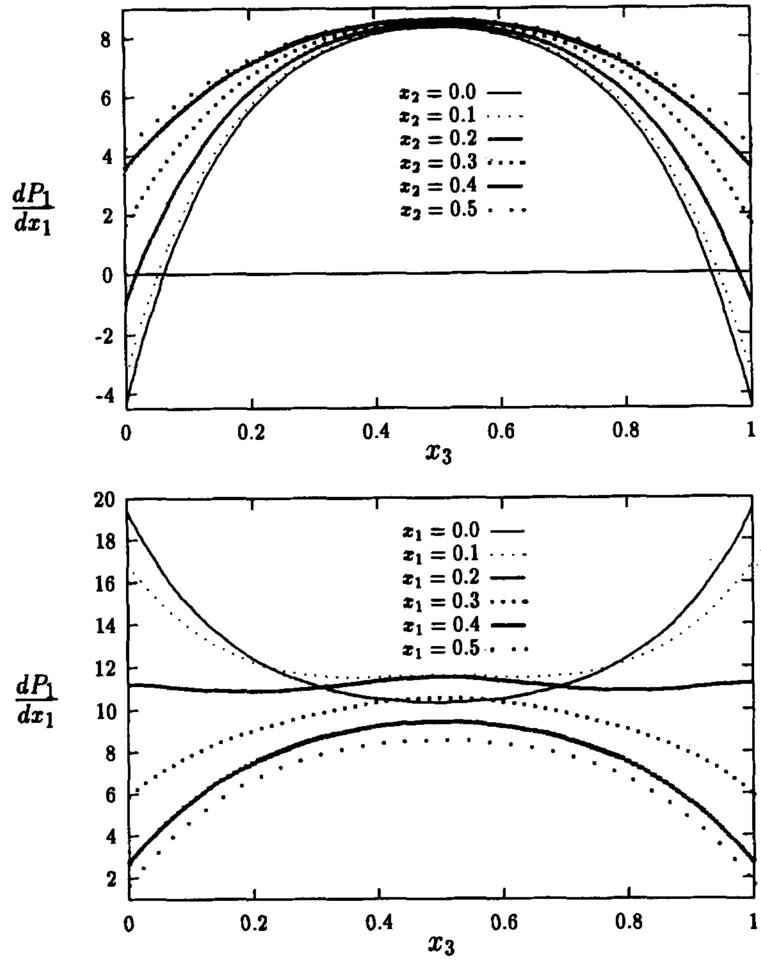


Fig. 9. The pressure as a function of x_3, for $a = b = H = 1$, $h = 0.5$. (a) $2\pi\partial P^1 \partial x_1$ for $x_1 = 0.5$ and several x_2; (b) $2\pi\partial P^1/\partial x_1$ for $x_2 = 0.3$ and several x_1.

when averaging in the x_2 direction, as in Liron [13], because then the integral over each of those graphs should be zero (zero flux). Thus we see that the local behavior changes drastically inside the basic cell unit, from all positive to all negative, say, as in Fig. 7a, and averaging first may lead to completely wrong results when used in applications. Other velocity components show similar behavior, see an example of U_2^2, in Fig. 8. In Figs. 9 we show distributions of the pressure gradient $\partial P^1/\partial x_1$.

5. Conclusions

We have discussed three problems of summing up, without averaging, an infinite number of identical periodically distributed stokeslets in three dimensions. The three cases are an infinite line above a flat plate, an infinite regular array above a flat plate and an infinite regular array between parallel plates. The last two cases were used originally in papers by the author, to model infinite arrays of cilia. In these works averaging in one space dimension was first performed. Here we obtained expressions for the velocity without any averaging. All expressions are rapidly converging infinite series of one form or another. The demonstration of the results indicate that for confined spaces the averaging first may lead to gross inaccuracies.

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Thanks are due to the Department of Mathematics at the University of Madison, Wisconsin, at which part of this work was completed while the author spent a Sabbatical there in 1994. Thanks are also due to Prof. S. Kim of the Chemical Engineering Department there for his financial support during the same period. K. Levit-Gurevich computed and drew the figures.

6. Appendix A

The full expressions for the velocity and pressure of an infinite line of stokeslets, as described in Section 2, is given below. We have replaced x_3 by z, $\alpha = z - h$, $\beta = z + h$, $s^2 = \alpha^2 + r_2^2$, $p^2 = \beta^2 + r_2^2$, see section 2. The following hold for $p, s > 0$.

$$8\pi\mu U_1^1 = \frac{4}{a}\ln\left(\frac{p}{s}\right) + \frac{8}{a}(T1) - \frac{32\pi^2 hz}{a^3}(T8) - \frac{8\pi}{a^2}(T2),$$

$$\begin{aligned} 8\pi\mu U_2^2 &= \frac{2}{a}\ln\left(\frac{p}{s}\right) + \frac{4}{a}(T1) + \frac{2r_2^2}{a}\left(\frac{1}{s^2} - \frac{1}{p^2}\right) + \frac{8\pi}{a^2}r_2^2(T5) - \frac{4hz}{ap^2} - \frac{16\pi hz}{a^2 p}(T11) \\ &+ \frac{8r_2^2 hz}{p^4 a} + \frac{32\pi^2 hzr_2^2}{a^3 p^2}(T8) + \frac{32\pi hzr_2^2}{a^2 p^3}(T11), \end{aligned}$$

$$\begin{aligned} 8\pi\mu U_3^3 &= \frac{2}{a}\ln\left(\frac{p}{s}\right) + \frac{4}{a}(T1) + \frac{2}{a}\left(\frac{\alpha^2}{s^2} - \frac{\beta^2}{p^2}\right) + \frac{8\pi}{a^2}(T4) + \frac{4hz}{p^2 a} + \frac{16\pi hz}{a^2 p}(T11) \\ &- \frac{8\beta^2 hz}{p^4 a} - \frac{32\pi^2 hz\beta^2}{a^3 p^2}(T8) - \frac{32\pi hz\beta^2}{a^2 p^3}(T11), \end{aligned}$$

$$8\pi\mu U_2^1 = \frac{8\pi r_2^2}{a^2}(T3) + \frac{32\pi^2 hzr_2}{a^3 p}(T9) = 8\pi\mu U_1^2,$$

$$8\pi\mu U_3^1 = \frac{8\pi}{a^2}(T6) + \frac{16\pi h}{a^2}(T10) + \frac{32\pi^2 hz\beta}{a^3 p}(T9),$$

$$\begin{aligned} 8\pi\mu U_3^2 &= \frac{2r_2}{a}\left(\frac{\alpha}{s^2} - \frac{\beta}{p^2}\right) + \frac{8\pi r_2}{a^2}(T7) + \frac{4hr_2}{ap^2} + \frac{16\pi hr_2}{a^2 p}(T11) \\ &+ \frac{8hz\beta r_2}{p^4 a} + \frac{32\pi^2 hz\beta r_2}{a^3 p^2}(T8) + \frac{32\pi hz\beta r_2}{a^2 p^3}(T11), \end{aligned}$$

$$8\pi\mu U_1^3 = \frac{8\pi}{a^2}(T6) + \frac{16\pi h}{a^2}(T10) - \frac{32\pi^2 h\alpha\beta}{a^2 p}(T9),$$

$$\begin{aligned} 8\pi\mu U_2^3 &= \frac{2r_2}{a}\left(\frac{\alpha}{s^2} - \frac{\beta}{p^2}\right) + \frac{8\pi r_2}{a^2}(T7) + \frac{4hr_2}{ap^2} + \frac{16\pi hr_2}{a^2 p}(T11) \\ &- \frac{8hz\beta r_2}{p^4 a} - \frac{32\pi^2 hz\beta r_2}{a^3 p^2}(T8) - \frac{32\pi hz\beta r_2}{a^2 p^3}(T11). \end{aligned}$$

The corresponding pressures for $\mathbf{U}^k$, $k = 1, 2, 3$, are

$$4\pi P^1 = \frac{8\pi}{a^2}(T3) + \frac{32\pi^2 h\beta}{a^3 p}(T9),$$

$$\begin{aligned}4\pi P^2 &= \frac{2r_2^2}{a}\left(\frac{1}{s^2} - \frac{1}{p^2}\right) + \frac{8\pi r_2}{a^2}(T5) + \frac{8h\beta r_2}{p^4 a} \\ &+ \frac{32\pi^2 h\beta r_2}{a^3 p^2}(T8) + \frac{32\pi h\beta r_2}{a^2 p^3}(T11),\end{aligned}$$

$$\begin{aligned}4\pi P^3 &= \frac{2}{a}\left(\frac{\alpha}{s^2} - \frac{\beta}{p^2}\right) + \frac{8\pi}{a^2}(T7) + \frac{4h}{p^2 a} + \frac{16\pi h}{a^2 p}(T11) - \frac{8h\beta^2}{p^4 a} \\ &- \frac{32\pi^2 h\beta^2}{a^3 p^2}(T8) - \frac{32\pi h\beta^2}{a^2 p^3}(T11).\end{aligned}$$

The T's appearing above are the following infinite series;

$$(T1) = \sum_{l=1}^{\infty} \cos\left(\frac{2\pi l r_1^0}{a}\right)\left[K_0\left(\frac{2\pi l s}{a}\right) - K_0\left(\frac{2\pi l p}{a}\right)\right],$$

$$(T2) = \sum_{l=1}^{\infty} l\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[sK_1\left(\frac{2\pi l s}{a}\right) - pK_1\left(\frac{2\pi l p}{a}\right)\right],$$

$$(T3) = \sum_{l=1}^{\infty} l\sin\left(\frac{2\pi l r_1^0}{a}\right)\left[K_1\left(\frac{2\pi l s}{a}\right) - K_0\left(\frac{2\pi l p}{a}\right)\right],$$

$$(T4) = \sum_{l=1}^{\infty} l\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[\frac{\alpha^2}{s}K_1\left(\frac{2\pi l s}{a}\right) - \frac{\beta^2}{p}K_1\left(\frac{2\pi l p}{a}\right)\right],$$

$$(T5) = \sum_{l=1}^{\infty} l\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[\frac{1}{s}K_1\left(\frac{2\pi l s}{a}\right) - \frac{1}{p}K_1\left(\frac{2\pi l p}{a}\right)\right],$$

$$(T6) = \sum_{l=1}^{\infty} l\sin\left(\frac{2\pi l r_1^0}{a}\right)\left[\alpha K_0\left(\frac{2\pi l s}{a}\right) - \beta K_0\left(\frac{2\pi l p}{a}\right)\right],$$

$$(T7) = \sum_{l=1}^{\infty} l\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[\frac{\alpha}{s}K_1\left(\frac{2\pi l s}{a}\right) - \frac{\beta}{p}K_1\left(\frac{2\pi l p}{a}\right)\right],$$

$$(T8) = \sum_{l=1}^{\infty} l^2\cos\left(\frac{2\pi l r_1^0}{a}\right)K_0\left(\frac{2\pi l p}{a}\right),$$

$$(T9) = \sum_{l=1}^{\infty} l^2\sin\left(\frac{2\pi l r_1^0}{a}\right)K_1\left(\frac{2\pi l p}{a}\right),$$

$$(T10) = \sum_{l=1}^{\infty} l\sin\left(\frac{2\pi l r_1^0}{a}\right)K_0\left(\frac{2\pi l p}{a}\right),$$

$$(T11) = \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) K_1\left(\frac{2\pi l p}{a}\right).$$

Here K_0 and K_1 are the modified Bessel functions of the first kind of order 0 and 1 respectively, see Abramowitz & Stegun [16].

Appendix B

The full expressions for the velocity and pressure of a doubly infinite array of stokeslets above a flat plate, as described in Section 3. We have replaced x_3 by z, $\alpha = z - h$, $\beta = z + h$. Alternative expressions are given to cover all possibilities. The bracketed numbers stand for infinite series and closed expressions, see below. As before K stands for the modified Bessel function.

Velocity components

$$\begin{aligned} 8\pi\mu U_1^1 &= \frac{1}{b}(2) + \frac{4}{b}(8) - \frac{\pi}{ab}(5) + \frac{8\pi}{b^2}(43) - \frac{2\pi^2 hz}{a^2 b}(33) + \frac{16\pi hz}{b^2}(51) \\ &+ \frac{32\pi^2 hz}{b^3}(20) - \frac{32\pi^3 hz\beta^2}{b^3}(53) - \frac{32\pi hz\beta^2}{b^2}(52), (\alpha^2 + (r_1^0)^2 > 0), \end{aligned}$$

$$8\pi\mu U_1^1 = \frac{2}{a}(1) + \frac{8}{a}(7) - \frac{32\pi^2 hz}{a^3}(9) - \frac{8\pi}{a^2}(10), (\alpha^2 + (r_2^0)^2 > 0),$$

$$8\pi\mu U_2^2 = \frac{2}{b}(2) + \frac{8}{b}(8) - \frac{32\pi^2 hz}{b^3}(20) - \frac{8\pi}{b^2}(19), (\alpha^2 + (r_1^0)^2 > 0),$$

$$\begin{aligned} 8\pi\mu U_2^2 &= \frac{1}{a}(1) + \frac{4}{a}(7) - \frac{\pi}{ab}(6) + \frac{8\pi}{a^2}(44) - \frac{2\pi^2 hz}{b^2 a}(46) + \frac{16\pi hz}{a^2}(54) \\ &+ \frac{32\pi^2 hz}{a^3}(9) - \frac{32\pi^3 hz\beta^2}{a^3}(56) - \frac{32\pi hz\beta^2}{a^2}(55), (\alpha^2 + (r_2^0)^2 > 0), \end{aligned}$$

$$\begin{aligned} 8\pi\mu U_3^3 &= \frac{1}{b}(2) + \frac{4}{b}(8) + \frac{\pi}{ab}(5) + \frac{8\pi}{b^2}(25) + \frac{2\pi^2 hz}{a^2 b}(27) + \frac{16\pi hz}{b^2}(51) \\ &- \frac{32\pi^2 hz\beta^2}{b^3}(53) - \frac{32\pi hz\beta^2}{b^2}(52), (\alpha^2 + (r_1^0)^2 > 0), \end{aligned}$$

$$\begin{aligned} 8\pi\mu U_3^3 &= \frac{1}{a}(1) + \frac{4}{a}(7) + \frac{\pi}{ab}(6) + \frac{8\pi}{a^2}(26) + \frac{2\pi^2 hz}{ab^2}(28) + \frac{16\pi hz}{b^2}(54) \\ &- \frac{32\pi^2 hz\beta^2}{a^3}(56) - \frac{32\pi hz\beta^2}{a^2}(55), (\alpha^2 + (r_2^0)^2 > 0), \end{aligned}$$

$$8\pi\mu U_2^1 = \frac{8\pi}{b^2}(11) + \frac{32\pi^2 hz}{b^3}(13) = 8\pi\mu U_1^2, (\alpha^2 + (r_1^0)^2 > 0),$$

$$8\pi\mu U_2^1 = \frac{8\pi}{a^2}(12) + \frac{32\pi^2 hz}{a^3}(18) = 8\pi\mu U_1^2, (\alpha^2 + (r_2^0)^2 > 0),$$

$$\begin{aligned} 8\pi\mu U_3^1 &= \frac{\pi}{ab}(4) + \frac{8\pi}{b^2}(15) + \frac{2\pi h}{ab}(34) + \frac{16\pi h}{b^2}(57) + \frac{2\pi^2 hz}{a^2 b}(45) \\ &+ \frac{32\pi^2 hz\beta}{b^3}(49) + \frac{32\pi hz\beta}{b^2}(52), (\alpha^2 + (r_1^0)^2 > 0), \end{aligned}$$

$$8\pi\mu U_3^1 = \frac{8\pi}{a^2}(14) + \frac{16\pi h}{a^2}(17) + \frac{32\pi^2 hz\beta}{a^3}(16), (\alpha^2 + (r_2^0)^2 > 0),$$

$$8\pi\mu U_3^2 = \frac{8\pi}{b^2}(21) + \frac{16\pi h}{b^2}(23) + \frac{32\pi^2 hz\beta}{b^3}(22), (\alpha^2 + (r_1^0)^2 > 0),$$

$$\begin{aligned} 8\pi\mu U_3^2 &= \frac{\pi}{ab}(3) + \frac{8\pi}{a^2}(24) + \frac{2\pi h}{ab}(47) + \frac{16\pi h}{a^2}(50) + \frac{2\pi^2 hz}{b^2 a}(48) \\ &+ \frac{32\pi^2 hz\beta}{a^3}(42) + \frac{32\pi hz\beta}{a^2}(41), (\alpha^2 + (r_2^0)^2 > 0), \end{aligned}$$

$$\begin{aligned} 8\pi\mu U_1^3 &= \frac{\pi}{ab}(4) + \frac{8\pi}{b^2}(15) + \frac{2\pi h}{ab}(34) + \frac{16\pi h}{b^2}(57) - \frac{2\pi^2 hz}{a^2 b}(45) \\ &- \frac{32\pi^2 hz\beta}{b^3}(49) - \frac{32\pi hz\beta}{b^2}(52), (\alpha^2 + (r_1^0)^2 > 0), \end{aligned}$$

$$8\pi\mu U_1^3 = \frac{8\pi}{a^2}(14) + \frac{16\pi h}{a^2}(17) - \frac{32\pi^2 hz\beta}{a^3}(16), (\alpha^2 + (r_2^0)^2 > 0),$$

$$8\pi\mu U_2^3 = \frac{8\pi}{b^2}(21) + \frac{16\pi h}{b^2}(23) - \frac{32\pi^2 hz\beta}{b^3}(22), (\alpha^2 + (r_1^0)^2 > 0),$$

$$\begin{aligned} 8\pi\mu U_2^3 &= \frac{\pi}{ab}(3) + \frac{8\pi}{a^2}(24) + \frac{2\pi h}{ab}(47) + \frac{16\pi h}{a^2}(50) - \frac{2\pi^2 hz}{b^2 a}(48) \\ &- \frac{32\pi^2 hz\beta}{a^3}(42) - \frac{32\pi hz\beta}{a^2}(41), (\alpha^2 + (r_2^0)^2 > 0). \end{aligned}$$

Pressure

$$\begin{aligned} 4\pi P_1 &= \frac{\pi}{ab}(29) + \frac{8\pi}{b^2}(35) + \frac{2\pi^2 h}{a^2 b}(45) \\ &+ \frac{32\pi^2 h\beta}{b^3}(49) + \frac{32\pi h\beta}{b^2}(58), (\alpha^2 + (r_1^0)^2 > 0), \end{aligned}$$

$$4\pi P_1 = \frac{8\pi}{a^2}(31) + \frac{32\pi^2 h\beta}{a^3}(16), (\alpha^2 + (r_2^0)^2 > 0),$$

$$4\pi P_2 = \frac{8\pi}{b^2}(32) + \frac{32\pi^2 h\beta}{b^3}(22), (\alpha^2 + (r_1^0)^2 > 0),$$

$$\begin{aligned} 4\pi P_2 &= \frac{\pi}{ab}(30) + \frac{8\pi}{a^2}(36) + \frac{2\pi^2 h}{b^2 a}(48) \\ &+ \frac{32\pi^2 h\beta}{a^3}(42) + \frac{32\pi h\beta}{a^2}(41), (\alpha^2 + (r_2^0)^2 > 0), \end{aligned}$$

$$\begin{aligned} 4\pi P_3 &= \frac{\pi}{ab}(37) + \frac{8\pi}{b^2}(39) + \frac{2\pi^2 h}{a^2 b}(27) + \frac{16\pi h}{b^2}(51) \\ &- \frac{32\pi^2 h\beta}{b^3}(53) + \frac{32\pi h\beta^2}{b^2}(52), (\alpha^2 + (r_1^0)^2 > 0), \end{aligned}$$

$$\begin{aligned} 4\pi P_3 &= \frac{\pi}{ab}(38) + \frac{8\pi}{a^2}(40) + \frac{2\pi^2 h}{b^2 a}(28) + \frac{16\pi h}{a^2}(54) \\ &- \frac{32\pi^2 h\beta^2}{a^3}(56) - \frac{32\pi h\beta^2}{a^2}(55), (\alpha^2 + (r_2^0)^2 > 0). \end{aligned}$$

The bracketed numbers appear below. Whenever ρ_1 and ρ_2 appear in an expression and only ρ_1^2 is defined, then ρ_2 is obtained from ρ_1 by replacing α by β.

$$(1) = \ln\left[\frac{\sinh^2(\pi\beta/b)+\sin^2(\pi r_2^0/b)}{\sinh^2(\pi\alpha/b)+\sin^2(\pi r_2^0/b)}\right],$$

$$(2) = \ln\left[\frac{\sinh^2(\pi\beta/a)+\sin^2(\pi r_1^0/a)}{\sinh^2(\pi\alpha/a)+\sin^2(\pi r_1^0/a)}\right],$$

$$(3) = \frac{\alpha\sin(2\pi r_2^0/b)}{\sinh^2(\pi\alpha/b)+\sin^2(\pi r_2^0/b)} - \frac{\beta\sin(2\pi r_2^0/b)}{\sinh^2(\pi\beta/b)+\sin^2(\pi r_2^0/b)},$$

$$(4) = \frac{\alpha\sin(2\pi r_1^0/a)}{\sinh^2(\pi\alpha/a)+\sin^2(\pi r_1^0/a)} - \frac{\beta\sin(2\pi r_1^0/a)}{\sinh^2(\pi\beta/a)+\sin^2(\pi r_1^0/a)},$$

$$(5) = \frac{\alpha\sin(2\pi\alpha/a)}{\sinh^2(\pi\alpha/a)+\sin^2(\pi r_1^0/a)} - \frac{\beta\sinh(2\pi\beta/a)}{\sinh^2(\pi\beta/a)+\sin^2(\pi r_1^0/a)},$$

$$(6) = \frac{\alpha\sinh(2\pi\alpha/b)}{\sinh^2(\pi\alpha/b)+\sin^2(\pi r_2^0/b)} - \frac{\beta\sinh(2\pi\beta/b)}{\sinh^2(\pi\beta/b)+\sin^2(\pi r_2^0/b)},$$

$$(7) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[K_0\left(\frac{2\pi l\rho_1}{a}\right) - K_0\left(\frac{2\pi l\rho_2}{a}\right)\right], \rho_1^2 = \alpha^2 + (r_2^0+nb)^2,$$

$$(8) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}\cos\left(\frac{2\pi l r_2^0}{b}\right)\left[K_0\left(\frac{2\pi l\rho_1}{b}\right) - K_0\left(\frac{2\pi l\rho_2}{b}\right)\right], \rho_1^2 = \alpha^2 + (r_1^0+na)^2,$$

$$(9) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}l^2\cos\left(\frac{2\pi l r_1^0}{a}\right)K_0\left(\frac{2\pi l\rho_2}{a}\right), \rho_2^2 = \beta^2 + (r_2^0+nb)^2,$$

$$(10) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}l\cos\left(\frac{2\pi l r_1^0}{a}\right)\left[\rho_1 K_1\left(\frac{2\pi l\rho_1}{a}\right) - \rho_2 K_1\left(\frac{2\pi l\rho_2}{a}\right)\right], \rho_1^2 = \alpha^2 + (r_2^0+nb)^2,$$

$$(11) = \sum_{n=-\infty}^{\infty} r_1\sum_{l=1}^{\infty}l\sin\left(\frac{2\pi l r_2^0}{b}\right)\left[K_0\left(\frac{2\pi l\rho_1}{b}\right) - K_0\left(\frac{2\pi l\rho_2}{b}\right)\right], r_1 \equiv r_1^0 + na, \qquad \rho_1^2 = \alpha^2 + (r_1^0+na)^2,$$

$$(12) = \sum_{n=-\infty}^{\infty} r_2\sum_{l=1}^{\infty}l\sin\left(\frac{2\pi l r_1^0}{a}\right)\left[K_0\left(\frac{2\pi l\rho_1}{a}\right) - K_0\left(\frac{2\pi l\rho_2}{a}\right)\right], r_2 = r_2^0 + nb, \qquad \rho_1^2 = \alpha^2 + (r_2^0+nb)^2,$$

$$(13) = \sum_{n=-\infty}^{\infty}\frac{r_1}{\rho_2}\sum_{l=1}^{\infty}l^2\sin\left(\frac{2\pi l r_2^0}{b}\right)K_1\left(\frac{2\pi l\rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0+na)^2,$$

$$(14) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty}l\sin\left(\frac{2\pi l r_1^0}{a}\right)\left[\alpha K_0\left(\frac{2\pi l\rho_1}{a}\right) - \beta K_0\left(\frac{2\pi l\rho_2}{a}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0+nb)^2,$$

$$(15) = \sum_{n=-\infty}^{\infty} r_1\sum_{l=1}^{\infty}l\cos\left(\frac{2\pi l r_2^0}{b}\right)\left[\frac{\alpha}{\rho_1}K_1\left(\frac{2\pi l\rho_1}{b}\right) - \frac{\beta}{\rho_2}K_1\left(\frac{2\pi l\rho_2}{b}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_1^0+na)^2,$$

$$(16) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} \frac{l^2}{\rho_2} \sin\left(\frac{2\pi l r_1^0}{a}\right) K_1\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(17) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} 1 \sin\left(\frac{2\pi l r_1^0}{a}\right) K_0\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(18) = \sum_{n=-\infty}^{\infty} \frac{r_2}{\rho_2} \sum_{l=1}^{\infty} l^2 \sin\left(\frac{2\pi l r_1^0}{a}\right) K_1\left(\frac{2\pi l \rho_1}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(19) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_2^0}{b}\right) \left[\rho_1 K_1\left(\frac{2\pi l \rho_1}{b}\right) - \rho_2 K_1\left(\frac{2\pi l \rho_2}{b}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + nb)^2,$$

$$(20) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l^2 \cos\left(\frac{2\pi l r_2^0}{b}\right) K_0\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(21) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \sin\left(\frac{2\pi l r_2^0}{b}\right) \left[\alpha K_0\left(\frac{2\pi l \rho_1}{b}\right) - \beta K_0\left(\frac{2\pi l \rho_2}{b}\right)\right], \qquad \rho_2^2 = \alpha^2 + (r_1^0 + na)^2,$$

$$(22) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} \frac{l^2}{\rho_2} \sin\left(\frac{2\pi l r_1^0}{b}\right) K_1\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(23) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \sin\left(\frac{2\pi l r_2^0}{b}\right) K_0\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(24) = \sum_{n=-\infty}^{\infty} r_2 \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) \left[\frac{\alpha}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{a}\right) - \frac{\beta}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{a}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + nb)^2,$$

$$(25) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_2^0}{b}\right) \left[\frac{\alpha^2}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{b}\right) - \frac{\beta^2}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{b}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_1^0 + na)^2,$$

$$(26) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) \left[\frac{\alpha^2}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{a}\right) - \frac{\beta^2}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{a}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + nb)^2,$$

$$(27) = \frac{2\cosh(2\pi\beta/a)}{\sinh^2(\pi\beta\alpha/a) + \sin^2(\pi r_1^0/a)} - \frac{\sinh^2(2\pi\beta/a)}{[\sinh^2(\pi\beta/a) + \sin^2(\pi r_1^0/a)]^2},$$

$$(28) = \frac{2\cosh(2\pi\beta/b)}{\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b)} - \frac{\sinh^2(2\pi\beta/b)}{[\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b)]^2},$$

$$(29) = \frac{\sin(2\pi r_1^0/a)}{\sinh^2(\pi\alpha/a) + \sin^2(\pi r_1^0/a)} - \frac{\sin(2\pi r_1^0/a)}{\sinh^2(\pi\beta/a) + \sin^2(\pi r_1^0/a)},$$

$$(30) = \frac{\sin(2\pi r_2^0/b)}{\sinh^2(\pi\alpha/b) + \sin^2(\pi r_2^0/b)} - \frac{\sin(2\pi r_2^0/b)}{\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b)},$$

$$(31) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \sin\left(\frac{2\pi l r_1^0}{a}\right) \left[K_0\left(\frac{2\pi l \rho_1}{a}\right) - K_0\left(\frac{2\pi l \rho_2}{a}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + nb)^2,$$

$$(32) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \sin\left(\frac{2\pi l r_2^0}{b}\right) \left[K_0\left(\frac{2\pi l \rho_1}{b}\right) - K_0\left(\frac{2\pi l \rho_2}{b}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + na)^2,$$

$$(33) = \frac{2\cos(2\pi r_1^0/a)}{\sinh^2(\pi\beta/a) + \sin^2(\pi r_1^0/a)} - \left(\frac{\sin(2\pi r_1^0/a)}{\sinh^2(\pi\beta/a) + \sin^2(\pi r_1^0/a)}\right)^2,$$

$$(34) = \frac{\sin(2\pi r_1^0/a)}{\sinh^2(\pi\beta/a) + \sin^2(\pi r_1^0/a)},$$

$$(35) = \sum_{n=-\infty}^{\infty} r_1 \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_2^0}{b}\right) \left[\frac{1}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{b}\right) - \frac{1}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{b}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_1^0 + na)^2,$$

$$(36) = \sum_{n=-\infty}^{\infty} r_2 \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) \left[\frac{1}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{a}\right) - \frac{1}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{a}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + nb)^2,$$

$$(37) = \frac{\sinh(2\pi\alpha/a)}{\sinh^2(\pi\alpha/a) + \sin^2(\pi r_1^0/a)} - \frac{\sinh(2\pi\beta/a)}{\sinh^2(\pi\beta/a) + \sin^2(\pi r_1^0/a)},$$

$$(38) = \frac{\sinh(2\pi\alpha/b)}{\sinh^2(\pi\alpha/b) + \sin^2(\pi r_2^0/b)} - \frac{\sinh(2\pi\beta/b)}{\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b)},$$

$$(39) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_2^0}{b}\right) \left[\frac{\alpha}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{b}\right) - \frac{\beta}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{b}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_1^0 + na)^2,$$

$$(40) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) \left[\frac{\alpha}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{a}\right) - \frac{\beta}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{a}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + nb)^2,$$

$$(41) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) \frac{r_2}{\rho_2^3} K_1\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(42) = \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} l^2 \cos\left(\frac{2\pi l r_1^0}{a}\right) \frac{r_2}{\rho_2^2} K_1\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(43) = \sum_{n=-\infty}^{\infty} r_1^2 \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_2^0}{b}\right) \left[\frac{1}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{b}\right) - \frac{1}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{b}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_1^0 + na)^2,$$

$$(44) = \sum_{n=-\infty}^{\infty} r_2^2 \sum_{l=1}^{\infty} l \cos\left(\frac{2\pi l r_1^0}{a}\right) \left[\frac{1}{\rho_1} K_1\left(\frac{2\pi l \rho_1}{a}\right) - \frac{1}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{a}\right)\right], \qquad \rho_1^2 = \alpha^2 + (r_2^0 + nb)^2,$$

$$(45) = \frac{\sinh(2\pi\beta/a)\sin(2\pi r_1^0/a)}{(\sinh^2(\pi\beta/a) + \sin^2(\pi r_1^0/a))^2},$$

$$(46) = \frac{2\cos(2\pi r_2^0/b)}{\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b)} - \left(\frac{\sin(2\pi r_2^0/b)}{\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b)}\right)^2,$$

$$(47) = \frac{\sin(2\pi r_2^0/b)}{\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b)},$$

$$(48) = \frac{\sinh(2\pi\beta/b)\sin(2\pi r_2^0/b)}{(\sinh^2(\pi\beta/b) + \sin^2(\pi r_2^0/b))^2},$$

$$(49) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} l^2 \cos\left(\frac{2\pi l r_2^0}{b}\right)\frac{r_1}{\rho_2^2} K_0\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(50) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} 1 \cos\left(\frac{2\pi l r_1^0}{a}\right)\frac{r_2}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(51) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} 1 \cos\left(\frac{2\pi l r_2^0}{b}\right)\frac{1}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(52) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} 1 \cos\left(\frac{2\pi l r_2^0}{b}\right)\frac{1}{\rho_2^3} K_1\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(53) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} l^2 \cos\left(\frac{2\pi l r_2^0}{b}\right)\frac{1}{\rho_2^2} K_0\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(54) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} 1 \cos\left(\frac{2\pi l r_1^0}{a}\right)\frac{1}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(55) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} 1 \cos\left(\frac{2\pi l r_1^0}{a}\right)\frac{1}{\rho_2^3} K_1\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(56) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} l^2 \cos\left(\frac{2\pi l r_1^0}{a}\right) K_0\left(\frac{2\pi l \rho_2}{a}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + nb)^2,$$

$$(57) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} 1 \cos\left(\frac{2\pi l r_2^0}{b}\right)\frac{r_1}{\rho_2} K_1\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_1^0 + na)^2,$$

$$(58) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{\infty} 1 \cos\left(\frac{2\pi l r_2^0}{b}\right)\frac{r_1}{\rho_2^3} K_1\left(\frac{2\pi l \rho_2}{b}\right), \qquad \rho_2^2 = \beta^2 + (r_2^0 + na)^2.$$

Appendix C

The full expressions for the velocity and pressure of a doubly infinite array of stokeslets between two flat plates, as described in Section 4. Because of periodicity, it is sufficient to look at $\mathbf{x} = (x_1, x_2, x_3) = (r_1^0, r_2^0, x_3), 0 \le r_1^0 < a, 0 \le r_2^0 < b, 0 < x_3 < H$.

Velocity components

$$
\begin{aligned}
4\pi\mu U_1^1 &= \frac{\kappa_1\kappa_2}{\pi}\left\{\frac{h(H-x_3)}{H} + 2\sum_{m=1}^{\infty}\frac{f(\kappa_2 m)}{\kappa_2 m}\cos(\kappa_2 m r_2^0)\right.\\
&+\sum_{n=1}^{\infty}\left[\frac{f(\kappa_1 n)}{\kappa_1 n} + f'(\kappa_1 n)\right]\cos(\kappa_1 n r_1^0)\\
&+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{f(sq)}{sq} + \frac{(\kappa_1 n)^2}{(sq)^2}f'(sq) + \frac{(\kappa_2 m)^2}{(sq)^3}f(sq)\right]\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\\
&-\left[\frac{3h(H-h)x_3(H-x_3)}{H^3} + \sum_{n=1}^{\infty}(\kappa_1 n)A_1(\kappa_1 n)\cos(\kappa_1 n r_1^0)\right.\\
&\left.\left.+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\kappa_1 n)^2}{sq}A_1(sq)\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right]\right\}.
\end{aligned}
$$

For the definition of f, sq, etc., see below.

$$
\begin{aligned}
4\pi\mu U_2^2 &= \frac{\kappa_1\kappa_2}{\pi}\left\{\frac{h(H-x_3)}{H} + 2\sum_{n=1}^{\infty}\frac{f(\kappa_1 n)}{\kappa_1 n}\cos(\kappa_1 n r_1^0)\right.\\
&+\sum_{m=1}^{\infty}\left[\frac{f(\kappa_2 m)}{\kappa_2 m} + f'(\kappa_2 m)\right]\cos(\kappa_2 m r_2^0)\\
&+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{f(sq)}{sq} + \frac{(\kappa_2 m)^2}{(sq)^2}f'(sq) + \frac{(\kappa_1 n)^2}{(sq)^3}f(sq)\right]\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\\
&-\left[\frac{3h(H-h)x_3(H-x_3)}{H^3} + \sum_{m=1}^{\infty}(\kappa_2 m)A_1(\kappa_2 m)\cos(\kappa_2 m r_2^0)\right.\\
&\left.\left.+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\kappa_2 m)^2}{sq}A_1(sq)\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
4\pi\mu U_3^3 &= \frac{\kappa_1\kappa_2}{\pi}\left\{\sum_{m=1}^{\infty}\left[\frac{f(\kappa_2 m)}{\kappa_2 m} - f'(\kappa_2 m)\right]\cos(\kappa_2 m r_2^0)\right.\\
&+\sum_{n=1}^{\infty}\left[\frac{f(\kappa_1 n)}{\kappa_1 n} - f'(\kappa_1 n)\right]\cos(\kappa_1 n r_1^0)\\
&+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{f(sq)}{sq} - f'(sq)\right]\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\\
&+\sum_{n=1}^{\infty}\frac{A_2(\kappa_1 n)}{\kappa_1 n}\cos(\kappa_1 n r_1^0) + \sum_{m=1}^{\infty}\frac{A_2(\kappa_2 m)}{\kappa_2 m}\cos(\kappa_2 m r_2^0)\\
&\left.+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_2(sq)}{sq}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right\},
\end{aligned}
$$

$$
\begin{aligned}
&4\pi\mu U_{2,1}^{1,2}\\
&= \frac{2\kappa_1\kappa_2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(n\kappa_1)(m\kappa_2)\left[\frac{A_1(sq)}{sq} - \frac{f'(sq)}{(sq)^2} + \frac{f(sq)}{(sq)^3}\right]\sin(\kappa_1 n r_1^0)\sin(\kappa_2 m r_2^0),
\end{aligned}
$$

$$4\pi\mu U^{1,3}_{3,1} = \frac{\kappa_1\kappa_2}{\pi}\left\{\sum_{n=1}^{\infty}[(x_3-h)f(\kappa_1 n)+A_4(\kappa_1 n)]\sin(\kappa_1 n r_1^0)\right.$$
$$\left.+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\kappa_1 n)\left[(x_3-h)\frac{f(sq)}{sq}+\frac{A_4(sq)}{sq}\right]\sin(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right\},$$

$$4\pi\mu U^{2,3}_{3,2} = \frac{\kappa_1\kappa_2}{\pi}\left\{\sum_{m=1}^{\infty}[(x_3-h)f(\kappa_2 m)+A_4(\kappa_2 m)]\sin(\kappa_2 m r_2^0)\right.$$
$$\left.+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\kappa_2 m)\left[(x_3-h)\frac{f(sq)}{sq}+\frac{A_4(sq)}{sq}\right]\cos(\kappa_1 n r_1^0)\sin(\kappa_2 m r_2^0)\right\}.$$

Pressure

Together with the implicit condition that the total flux is zero for the velocities, it follows that there is a positive pressure head per basic length (a or b) in the x_1 and x_2 direction. Thus the *total pressures* P^1 or P^2 would be infinite, and we therefore look at $\partial P^1/\partial x_1$ and $\partial P^2/\partial x_2$. For P^3 the total pressure head is zero, and P^3 itself is given.

$$2\pi\frac{\partial P^1}{\partial x_1} = \frac{\kappa_1\kappa_2}{\pi}\left\{\sum_{n=1}^{\infty}(\kappa_1 n)f(\kappa_1 n)\cos(\kappa_1 n r_1^0)\right.$$
$$+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\kappa_1 n)^2\frac{f(sq)}{sq}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)$$
$$+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\kappa_1 n)^2\frac{A_5(sq)}{(sq)^2}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)$$
$$\left.+\frac{3h(H-h)}{H^3}+\sum_{n=1}^{\infty}(\kappa_1 n)A_5(\kappa_1 n)\cos(\kappa_1 n r_1^0)\right\},$$

$$2\pi\frac{\partial P^2}{\partial x_2} = \frac{\kappa_1\kappa_2}{\pi}\left\{\sum_{m=1}^{\infty}(\kappa_2 m)f(\kappa_2 m)\cos(\kappa_2 m r_2^0)\right.$$
$$+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\kappa_2 m)^2\frac{f(sq)}{sq}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)$$
$$+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(\kappa_2 m)^2\frac{A_5(sq)}{(sq)^2}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)$$
$$\left.+\frac{3h(H-h)}{H^3}+\sum_{m=1}^{\infty}(\kappa_1 m)A_5(\kappa_2 m)\cos(\kappa_2 m r_2^0)\right\},$$

$$2\pi P^3 = \frac{\kappa_1\kappa_2}{\pi}\left\{\operatorname{sgn}(x_3-h)\left[\frac{h}{2H}+\sum_{m=1}^{\infty}g(\kappa_2 m)\cos(\kappa_2 m r_2^0)\right.\right.$$
$$\left.+\sum_{n=1}^{\infty}g(\kappa_1 n)\cos(\kappa_1 n r_1^0)+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}g(sq)\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)\right]$$
$$+2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_6(sq)}{(sq)^2}\cos(\kappa_1 n r_1^0)\cos(\kappa_2 m r_2^0)+\sum_{n=1}^{\infty}\frac{A_6(\kappa_1 n)}{(\kappa_1 n)^2}\cos(\kappa_1 n r_1^0)$$

$$+\sum_{m=1}^{\infty}\frac{A_6(\kappa_2 m)}{(\kappa_2 m)^2}\cos(\kappa_2 m r_2^0)+\frac{h(H-h)(H-2h)}{2H^3}\Bigg\},$$

where:

$$\kappa_1=\frac{2\pi}{a};\kappa_2=\frac{2\pi}{b};sq=\sqrt{(\kappa_1 n)^2+(\kappa_2 m)^2}.$$

Define

$$\begin{aligned}\alpha(\lambda) &= \frac{1}{\sinh^2(\lambda H)-(\lambda H)^2}\{x_3[h\sinh(\lambda H)\cosh(\lambda(H-x_3-h))\\ &\quad -H\sinh(\lambda h)\cosh(\lambda x_3)]-Hh\sinh(\lambda x_3)\cosh(\lambda h)\\ &\quad +H^2\sinh(\lambda x_3)\sin(\lambda h)\coth(\lambda H)\},\end{aligned}$$

$$\begin{aligned}\beta(\lambda) &= -\frac{\lambda H}{\sinh^2(\lambda H)-(\lambda H)^2}\Big[x_3 h\cosh(\lambda(x_3-h))\\ &\quad +H(x_3+h)\sinh(\lambda h)\sinh(\lambda x_3)-H(x_3\sinh(\lambda h)\cosh(\lambda x_3)\\ &\quad +h\sinh(\lambda x_3)\cosh(\lambda h))\coth(\lambda H)+H^2\frac{\sinh(\lambda x_3)\sinh(\lambda h)}{\sinh^2(\lambda h)}\Big],\end{aligned}$$

then

$$\begin{aligned}A_1(\lambda) &= \alpha(\lambda)+\beta(\lambda)\\ A_2(\lambda) &= -\alpha(\lambda)+\beta(\lambda)\end{aligned}$$

$$\begin{aligned}A_4(\lambda) &= \frac{1}{\sinh^2(\lambda H)-(\lambda H)^2}\{\lambda h\, x_3\, H\sinh(\lambda(x_3-h))\\ &\quad +H^2\lambda\frac{[x_3\sinh(\lambda h)\sinh(\lambda(H-x_3))-h\sinh(\lambda x_3)\sinh(\lambda(H-h))]}{\sinh(\lambda H)}\\ &\quad \pm[x_3 H\sinh(\lambda x_3)\sinh(\lambda h)+x_3 h\sinh(\lambda(H-x_3-h))\sinh(\lambda H)\\ &\quad -H(H-h)\sinh(\lambda h)\sinh(\lambda x_3)]\},\end{aligned}$$

where the + sign is used for U_3^1, U_3^2, and the (−) sign is used for U_1^3, U_2^3.

$$\begin{aligned}A_5(\lambda) &= \frac{\lambda^2}{\sinh^2(\lambda H)-(\lambda H)^2}\{H\sinh(\lambda h)\sinh(\lambda x_3)\\ &\quad +h\sinh(\lambda H)\sinh(\lambda(h-x_3-h))+\lambda hH\sinh(\lambda(x_3-h))\\ &\quad +\lambda H^2\frac{\sinh(\lambda(H-x_3))\sinh(\lambda h)}{\sinh(\lambda H)}\Big\},\end{aligned}$$

$$\begin{aligned}A_6(\lambda) &= \frac{\lambda^2}{\sinh^2(\lambda H)-(\lambda H)^2}\{H\sinh(\lambda h)\cosh(\lambda x_3)\\ &\quad -h\sinh(\lambda H)\cosh(\lambda(H-x_3-h))\\ &\quad -\lambda H\left[h\cosh(\lambda(x_3-h))-H\frac{\cosh(\lambda(H-x_3))\sinh(\lambda h)}{\sinh(\lambda H)}\right]\Big\}.\end{aligned}$$

Symmetry properties of A_i:

$$\begin{aligned}A_1(H-x_3,H-h) &= A_1(x_3,h)\\ A_2(H-x_3,H-h) &= A_2(x_3,h)\\ A_4(H-x_3,H-h) &= -A_4(x_3,h)\\ A_5(H-x_3,H-h) &= A_5(x_3,h)\\ A_6(H-x_3,H-h) &= -A_6(x_3,h)\end{aligned}$$

$$f(\lambda) = \begin{cases} \dfrac{\sinh(\lambda h)\sinh(\lambda(H - x_3))}{\sinh(\lambda H)}, & x_3 > h \\ \dfrac{\sinh(\lambda(H - h))\sinh(\lambda x_3)}{\sinh(\lambda H)}, & x_3 < h \end{cases}$$

$$g(\lambda) = \begin{cases} \dfrac{\sinh(\lambda h)\cosh(\lambda(H - x_3))}{\sinh(\lambda H)}, & x_3 > h \\ \dfrac{\sinh(\lambda(H - h))\cosh(\lambda x_3)}{\sinh(\lambda H)}, & x_3 < h\,. \end{cases}$$

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